USC-05/HEP-B2 hep-th/0512091

Single twistor description of massless, massive, AdS, and other interacting particles

Itzhak Bars[†] **and Moises Picón**[†,‡]

[†] *Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089-0484, USA*

[‡] *Departamento de Física Téorica, Univ. de Valencia and IFIC (CSIC-UVEG), 46100-Burjassot (Valencia), Spain*

ABSTRACT

The Penrose transform between twistors and the phase space of massless particles is generalized from the massless case to an assortment of other particle dynamical systems, including special examples of massless or massive particles, relativistic or non-relativistic, interacting or non-interacting, in flat space or curved spaces. Our unified construction involves always the *same* twistor Z^A with only four complex degrees of freedom and subject to the *same* helicity constraint. Only the twistor to phase space transform differs from one case to another. Hence a unification of diverse particle dynamical systems is displayed by the fact that they all share the same twistor description. Our single twistor approach seems to be rather different and strikingly economical construction of twistors compared to other past approaches that introduced multiple twistors to represent some similar but far more limited set of particle phase space systems.

1 Introduction and concepts

Penrose introduced the twistor program as an alternative to phase space to give a description of relativistic massless particles in four dimensions [1][2]. The twistor description is not only Lorentz covariant, but is also $\mathrm{SU}(2,2)$ covariant and this makes evident the well known hidden conformal symmetry $\mathrm{SO}(4,2)=\mathrm{SU}(2,2)$ of the massless system. Penrose's twistors have turned out to be extremely useful to gain new insight in twistor string theory [3]-[8] and simplify practical computations in super Yang-Mills theory [9][10]. We are motivated by this success to develop a twistor description of more general particle dynamics which may play a similar useful role in some cases.

In this paper we will use a simple and unified construction of twistors that will generalize the Penrose transform from the massless particle case to an assortment of other particle dynamical systems, including special examples of massless or massive particles, relativistic or non-relativistic, interacting or non-interacting, in flat space or curved spaces. The basic formalism for relating phase space and twistor degrees

in freedom in any dimension was given in [11] and further developed in [8][12]. In this paper we apply it explicitly to examples in four dimensions that have dynamics beyond massless systems. Our unified construction involves always the *same* four degrees of freedom in the twistor $Z^A = \binom{\mu}{\lambda}$, $A = 1,2,3,4$, constructed from $\mathrm{SL}(2,C)$ doublet spinors $\mu^{\dot\alpha}, \lambda_\alpha$, each described by two complex degrees of freedom $\alpha, \dot\alpha = 1,2$, and subject to the *same* helicity constraint $Z^A \bar{Z}_A = \mu^{\dot\alpha}\bar\lambda_{\dot\alpha} + \lambda_\alpha \bar\mu^\alpha = 2h$, where h is the helicity of the particle[1]. Only the twistor to phase space transform, which generalizes the Penrose transform, is different in each one of our examples. This should be striking in view of previous attempts over many years to construct twistors for massive particles and other cases that introduced a doubling of the twistor degrees of freedom [2], [13]-[18]. Thus, our economical one-twistor construction, that also unifies a variety of cases, appears to be a breakthrough in the twistor realm.

We will show that the *same* twistor Z^A which is known to describe the phase space of an on-shell massless particle, is also equivalent to the phase space of all of the following systems

- the massless relativistic particle in $d = 4$ flat Minkowski space,
- the massive relativistic particle in $d = 4$ flat Minkowski space,
- the particle on AdS_4,
- the particle on $\mathrm{AdS}_3 \times \mathrm{S}^1$, $\mathrm{AdS}_2 \times \mathrm{S}^2$,
- the particle on $R \times S^3$,
- the nonrelativistic free particle in 3 space dimensions,
- the nonrelativistic hydrogen atom in 3 space dimensions,
- and a related family of other particle systems.

Sharing the same twistor equivalent naturally implies duality type relationships among these systems which may be surprising to some readers. Moreover, just like the twistor version of massless particles makes the hidden $\mathrm{SU}(2,2)$ conformal symmetry a linearly realized evident symmetry, the twistor versions of all the other particle systems listed above also make it evident that there is the *same* hidden $\mathrm{SU}(2,2)$ symmetry in each one of them. The interpretation of $\mathrm{SU}(2,2)$ is not conformal transformations of phase space except for the massless case, but for all cases $\mathrm{SU}(2,2)=\mathrm{SO}(4,2)$ is a hidden symmetry that is closely related to an underlying $4+2$ dimensional flat spacetime, and realized in $3+1$ dimensional phase space in different non-linear ways for each of the systems listed above. In terms of the twistors, the generators of $\mathrm{SU}(2,2)$

[1] At the quantum level the ordered product $\frac{1}{2}(Z^A \bar{Z}_A + \bar{Z}_A Z^A)\psi = 2h\psi$ is applied on wavefunctions in twistor space. Using $\bar{Z}_A \psi = -\frac{\partial \psi}{\partial Z^A}\psi$, this produces Penrose's homogeneity constraints $Z^A \frac{\partial}{\partial Z^A}\psi(Z) = (-2h-2)\,\psi(Z)$.

are just quadratic and given by the 4×4 matrix $J^A_{\ B} = Z^A\bar{Z}_B - \frac{1}{4}\delta^A_{\ B}\ \left(Z^C\bar{Z}_C\right)$ which may be expanded in terms of the $\mathrm{SO}(4,2)$ gamma matrices Γ^{MN}

$$J^A_{\ B} = Z^A\bar{Z}_B - \frac{1}{4}\delta^A_{\ B}\ \left(Z^C\bar{Z}_C\right) = \frac{1}{4i}L_{MN}\ \left(\Gamma^{MN}\right)^A_{\ B}. \tag{1}$$

The $J^A_{\ B}$ satisfy the $\mathrm{SU}(2,2)$ Lie algebra when the twistors satisfy the canonical commutation rules $\left[Z^A,\bar{Z}_B\right] = \delta^A_{\ B}$. When the twistors Z^A are expressed in terms of the various phase space degrees of freedom of the systems above, we will show that the $\mathrm{SO}(4,2)$ generators L_{MN} take various non-linear phase space forms and close correctly to form the $\mathrm{SO}(4,2)$ Lie algebra under the phase space Poisson brackets at the classical level[2].

In this paper we will discuss mainly the zero helicity case $h=0$. For general helicity we use precisely the same twistor Z^A, but the twistor transform is slightly different due to the helicity, as discussed in detail elsewhere [19].

Our results on the hidden $\mathrm{SU}(2,2)$ may seem conceptually surprising independent of twistors: how can a massive particle have $\mathrm{SO}(4,2)$ symmetry? isn't the symmetry of AdS_4 given by $\mathrm{SO}(3,2)$ rather than $\mathrm{SO}(4,2)$? similarly isn't the symmetry of $\mathrm{AdS}_2\times \mathrm{S}^2$ given by $\mathrm{SO}(1,2)\times\mathrm{SO}(3)$ rather than SO(4,2), etc.? The larger unexpected hidden symmetry $\mathrm{SO}(4,2)$ is indeed difficult to notice, but its origin was already naturally explained in the context of two-time physics (2T-physics) [20][21]. Indeed our approach in the current paper uses directly previous results of 2T-physics on the phase spaces of the 1T systems above. We are led to our twistor results by relating the twistor gauge of 2T-physics to other gauge choices in phase space. Therefore from the point of view of 2T-physics, the hidden symmetries which become the evident $\mathrm{SU}(2,2)$ acting on the 4-component twistor Z^A are no surprise: This is the same as the $\mathrm{SO}(4,2)$ acting linearly on six dimensional phase space X^M, P^M in 2T-physics.

We will give only a very brief description of the concepts of 2T-physics, and then use it as a technique to construct the relations between the various phase spaces and the twistors Z^A. A particle in 1T-physics, interacting with various backgrounds in $(d-1)+1$ dimensions (e.g. electromagnetism, gravity, high spin fields, any potential, etc.), can be equivalently described in 2T-physics. The 2T theory is in d+2 dimensions, but has enough gauge symmetry to compensate for the extra 1+1 dimensions, so that the physical (gauge invariant) degrees of freedom are equivalent to those encountered in one-time physics (1T-physics). The general 2T theory for a particle moving in any background field has been constructed [22]

$$S = \int d\tau\ \left(\dot{X}^M P_M - \frac{1}{2}A^{ij}Q_{ij}\left(X,P\right)\right), \tag{2}$$

where the symmetric A^{ij} , $i,j=1,2$, is the $\mathrm{Sp}(2,R)$ gauge field, and the three $Q_{ij}\left(X,P\right)$ which depend on background fields, are required to form an $\mathrm{Sp}(2,R)$ algebra. The background fields must satisfy certain conditions to comply with the

[2] After quantum ordering nonlinear phase space factors, the L_{MN} satisfy the correct $\mathrm{SO}(4,2)$ Lie algebra at the quantum level, as demonstrated for most of these systems elsewhere [20][21]. In this paper we will only discuss the classical level.

Sp$(2,R)$ requirement. An infinite number of solutions to the requirement can be constructed [22]. So any 1T particle worldline theory, with any backgrounds, can be obtained as a gauge fixed version of some 2T particle worldline theory.

The fundamental gauge symmetry in 2T-physics is Sp(2,R) acting on phase space. A consequence of this gauge symmetry is that position and momentum become indistinguishable at any instant, so the symmetry is of fundamental significance. The transformation of X^M, P_M is generally a nonlinear map that can be explicitly given in the presence of background fields [22], but in the absence of backgrounds the transformation reduces to a linear doublet action of Sp$(2,R)$ on $\left(X^M, P^M\right)$ for each M [20]. The physical phase space is the subspace that is gauge invariant under Sp$(2,R)$. The gauge invariant subspace of $d+2$ dimensional phase space X^M, P_M is a phase space in $(d-1)+1$ dimensions x^μ, p_μ. However there are many possible ways to embed the $(d-1)+1$ phase space in $d+2$ phase space, and this is done by making Sp(2,R) gauge choices. In the resulting gauge fixed 1T system, time, Hamiltonian, and in general curved spacetime, are emergent concepts. The Hamiltonian, and therefore the dynamics as tracked by the emergent time, may look quite different in one gauge versus another gauge in terms of the remaining gauge fixed degrees of freedom. In this way, a unique 2T-physics action gives rise to many 1T-physics dynamical systems.

So, one of the strikingly surprising aspects of 2T physics is that the $d+2$ dimensional theory has many holographic images in $(d-1)+1$ dimensions. Each image fully captures the gauge invariant physical content of a unique parent theory, but from the point of view of 1T-physics each image appears as a different 1T-dynamical system. Thus 2T-physics unifies many 1T-systems into a family that corresponds to a given 2T-physics parent in $d+2$ dimensions. The members of such a family naturally must obey duality-type relationships among them and share many common properties, such as the same overall global symmetry that may be manifested in hidden non-linear ways on the fewer $(d-1)+1$ dimensions.

Thus, 2T-physics can be viewed as a unification approach through higher dimensions, but distinctly different than Kaluza-Klein theory because there are no Kaluza-Klein towers of states, but instead there is a family of 1T systems with duality type relationships among them.

The 1T systems on our list above are members of such a family for $d=4$. In the present case the parent 2T theory is the simplest version of 2T-physics without any background fields. The 2T action is [20]

$$S=\frac{1}{2}\int d\tau\ D_\tau X_i^M X_j^N \eta_{MN}\varepsilon^{ij}=\int d\tau\ \left(\dot{X}^M P^N-\frac{1}{2}A^{ij}X_i^M X_j^N\right)\eta_{MN}. \tag{3}$$

Here $X_i^M=\left(X^M\ P^M\right)$, $i=1,2$, is a doublet under Sp$(2,R)$ for every M, the structure $D_\tau X_i^M=\partial_\tau X_i^M-A_i{}^j X_j^M$ is the Sp(2,R) gauge covariant derivative, Sp(2,R) indices are raised and lowered with the antisymmetric Sp$(2,R)$ metric ε^{ij}, and in the last expression an irrelevant total derivative $-(1/2)\,\partial_\tau\,(X\cdot P)$ is dropped from the action. This action describes a particle that obeys the Sp(2,R) gauge symmetry, so its momentum and position are locally indistinguishable due to the gauge symmetry. The Sp$(2,R)$ constraints $Q_{ij}=X_i\cdot X_j=0$ that follow from the equations of motion

have non-trivial solutions only if the metric η_{MN} has two timelike dimensions. So for a system in which position and momentum are locally indistinguishable, to be non-trivial, two timelike dimensions are necessary as a consequence of the $\mathrm{Sp}(2,R)$ gauge symmetry.

Thus in our case the target space is flat in $4+2$ dimension, and hence this system has an $\mathrm{SO}(4,2)$ global symmetry. This global symmetry is shared in the same representation by all the emergent lower dimensional theories in the same family, and this explains the hidden $\mathrm{SO}(4,2){=}\mathrm{SU}(2,2)$ that is present in all the systems on our list above. Although this is an already established fact in previous work on 2T-physics, it will come into a new focus by displaying the explicit twistor/phase space transforms given in this paper. This will be achieved by using the fact that the twistor description in terms of the 4-component Z^A is one of the possible gauge choices of the same theory [11][8][12]. From the point of view of twistors this is a striking new perspective for constructing the twistor equivalent of particle dynamics. The same method has been applied also in higher dimensions $d=6,10,11$ and produced twistor equivalents for several interesting spaces, including supersymmetric $\mathrm{AdS}_5\times\mathrm{S}^5$, $\mathrm{AdS}_4\times\mathrm{S}^7$, $\mathrm{AdS}_7\times\mathrm{S}^4$ and others [23][7][8].

It is useful to do some simple counting of degrees of freedom. The on shell phase space $x^{\mu}(\tau), p^{\mu}(\tau)$ for a massless particle in four spacetime dimensions, after eliminating one gauge degree of freedom due to reparametrization of the worldline, and solving the mass-shell constraint $p^2=0$, has 3 independent positions and 3 independent momentum degrees of freedom. The complex four component twistor space Z^A also has exactly 6 real physical degrees of freedom after eliminating one overall phase from Z^A and solving one real constraint $Z^A\bar{Z}_A=2h$. The counting of the phase space physical degrees of freedom for any particle moving in 3+1 dimensions is also precisely 6, independent of whether the particle is massless, massive, moving in flat space or moving in curved space, or interacting with some potential. Given that the systems listed above all have 6 physical degrees of freedom, and are already identified in 2T-physics as being holographic representatives of a unique 4+2 dimensional system, we should expect that they all must be represented by the same SU(2,2) twistors. This is the idea that we explore in this paper in order to construct explicitly the relation between one unique set of twistors Z^A and the phase space for each of the systems listed above.

2 Twistor space

The twistor space $Z^A=\binom{\mu}{\lambda}$ is classified as the fundamental representation of $\mathrm{SU}(2,2)$, $A=1,2,3,4$. It consists of four complex numbers $\mu^{\dot{\alpha}}$, $\dot{\alpha}=1,2$, and λ_{α}, $\alpha=1,2$ which are the two spinor representations of the Lorentz group $\mathrm{SL}(2,C)$. One defines $\bar{Z}_A=\left(\bar{\lambda}_{\dot{\alpha}}\ \bar{\mu}^{\alpha}\right)=Z^{\dagger}\eta$, where $\eta=\tau_1\times 1$ is the $\mathrm{SU}(2,2)$ metric, and an overbar such as $\bar{\lambda}_{\dot{\alpha}}$ means complex conjugation of λ_{α}. The Z^A are identified up to an overall phase $Z^A\sim e^{i\phi}Z^A$, and satisfy a constraint $Z^A\bar{Z}_A=2h$, where the constant h is the helicity of the particle. The irrelevant phase together with the constraint remove two real degrees of freedom, so that the twistor contains 6 real physical degrees of freedom.

We will show that these are equivalent to the six physical degrees of freedom of the phase space of a particle in each of the cases in our list.

In the case of the massless particle the (twistor) $\leftrightarrow$ (phase space) equivalence is encoded in the Penrose relations (written below for a helicity zero particle $h=0$ that satisfies $Z^A\bar{Z}_A=0$ at the classical level)

$$\mu^{\dot{\alpha}}=-ix^{\dot{\alpha}\beta}\lambda_\beta,\ \lambda_\alpha\bar{\lambda}_{\dot{\beta}}=p_{\alpha\dot{\beta}}, \tag{4}$$

where the 2×2 Hermitian matrices $x^{\dot{\alpha}\beta}$, $p_{\alpha\dot{\beta}}$ are expanded in terms of the Pauli matrices

$$x^{\dot{\alpha}\beta}\equiv\frac{1}{\sqrt{2}}x^\mu\left(\bar{\sigma}_\mu\right)^{\dot{\alpha}\beta},\ p_{\alpha\dot{\beta}}\equiv\frac{1}{\sqrt{2}}p^\mu\left(\sigma_\mu\right)_{\alpha\dot{\beta}};\ \sigma_\mu\equiv\left(1,\vec{\sigma}\right),\ \bar{\sigma}_\mu\equiv\left(-1,\vec{\sigma}\right). \tag{5}$$

So, roughly speaking, λ is the square root of momentum p, while position x is the ratio μ/λ. We will give below the analogous (twistor) $\leftrightarrow$ (phase space) equivalence relations for the other cases in our list.

The properties of the twistors Z^A can be derived from the following τ reparametrization invariant action on the worldline

$$S=\int d\tau\left(i\bar{Z}_ADZ^A-2hV\right),\quad DZ^A\equiv\frac{\partial Z^A}{\partial\tau}-iVZ^A. \tag{6}$$

Here the 1-form $Vd\tau$ is a U(1) gauge field on the worldline, DZ^A is the gauge covariant derivative that satisfies $\delta_\varepsilon\left(DZ^A\right)=i\varepsilon\left(DZ^A\right)$ for $\delta_\varepsilon V=\partial\varepsilon/\partial\tau$ and $\delta_\varepsilon Z^A=i\varepsilon Z^A$. Note that the term $2hV$ (absent in previous literature) is gauge invariant since it transforms as a total derivative under the infinitesimal gauge transformation. The reason for requiring the U(1) gauge symmetry is the fact that the overall phase of the Z^A is unphysical and drops out in the relation between phase space and twistors, as in Eq.(4). Furthermore, the equation of motion with respect to V imposes the constraint $Z^A\bar{Z}_A-2h=0$, which is interpreted as the helicity constraint. Taking into account that $\left(Z^A\bar{Z}_A-2h\right)$ is the generator of the U(1) gauge transformations, the meaning of the vanishing generator (or helicity constraint) is that only the U(1) gauge invariant sector of twistor space is physical.

It is a small exercise to show that the twistor defined by this action, together with the Penrose relation in Eq.(4) do indeed correctly describe the phase space of the massless and spinless ($h=0$ at the classical level) relativistic particle. First, thanks to $\mu^{\dot{\alpha}}=-ix^{\dot{\alpha}\beta}\lambda_\alpha$, the constraint is explicitly satisfied at the classical level

$$\bar{Z}_AZ^A=\left(\bar{\lambda}_{\dot{\alpha}}\ \bar{\mu}^\alpha\right)\begin{pmatrix}\mu^{\dot{\alpha}}\\ \lambda_\alpha\end{pmatrix}=\bar{\lambda}_{\dot{\alpha}}\mu^{\dot{\alpha}}+\bar{\mu}^\alpha\lambda_\alpha=-i\bar{\lambda}_{\dot{\alpha}}x^{\dot{\alpha}\beta}\lambda_\beta+i\bar{\lambda}_{\dot{\beta}}x^{\dot{\beta}\alpha}\lambda_\alpha=0. \tag{7}$$

Second, the remaining term in the action $S_0=i\int d\tau\bar{Z}_A\frac{\partial Z^A}{\partial\tau}=i\int d\tau\left(\bar{\lambda}_{\dot{\alpha}}\frac{\partial\mu^{\dot{\alpha}}}{\partial\tau}+\ \bar{\mu}^\alpha\frac{\partial\lambda_\alpha}{\partial\tau}\right)$ that defines the canonical structure for the twistors $\left[Z^A,\bar{Z}_B\right]=\delta^A_B$ also correctly defines the canonical structure for the phase space variables x^μ,p_μ by substituting

$\mu^{\dot{\alpha}} = -ix^{\dot{\alpha}\beta}\lambda_\alpha$ as follows

$$S_0 = \int d\tau \left[\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial\tau}\left(x^{\dot{\alpha}\beta}\lambda_\beta\right) - \bar{\lambda}_{\dot{\alpha}} x^{\dot{\alpha}\beta}\frac{\partial\lambda_\beta}{\partial\tau}\right] = \int d\tau \frac{\partial x^{\dot{\alpha}\beta}}{\partial\tau}\lambda_\beta\bar{\lambda}_{\dot{\alpha}} \tag{8}$$

$$= \frac{1}{2}\int d\tau \frac{\partial x_\mu}{\partial\tau} p_\nu Tr\left(\bar{\sigma}^\mu\sigma^\nu\right) = \int d\tau \frac{\partial x_\mu}{\partial\tau} p_\mu. \tag{9}$$

This indicates that, aside from factor ordering issues, quantization of phase space $[x^\mu, p_\nu] = i\delta^\mu{}_\nu$ is consistent with quantization of twistor space $\left[Z^A, \bar{Z}_B\right] = \delta^A{}_B$. Finally, the form $p_{\alpha\dot{\beta}} = \lambda_\alpha\bar{\lambda}_{\dot{\beta}} = \frac{1}{\sqrt{2}}p_\mu\left(\sigma^\mu\right)_{\alpha\dot{\beta}}$ automatically satisfies the mass shell condition $p^\mu p_\mu = 0$ for a massless particle with positive energy $p_0 > 0$, as seen explicitly by writing out the matrix form

$$p_{\alpha\dot{\beta}} = \begin{pmatrix} \lambda_1\lambda_1^* & \lambda_1\lambda_2^* \\ \lambda_2\lambda_1^* & \lambda_2\lambda_2^* \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} p^0 + p_3 & p_1 - ip_2 \\ p + ip_2 & p^0 - p_3 \end{pmatrix} \tag{10}$$

$$Tr\frac{p_{\alpha\dot{\beta}}}{\sqrt{2}} = \frac{|\lambda_1|^2 + |\lambda_{21}|^2}{\sqrt{2}} = p_0, \ \det\left(p_{\alpha\dot{\beta}}\right) = 0 = \frac{1}{2}p^\mu p_\mu. \tag{11}$$

In the remainder of this paper we will show that the same twistor space Z^A described by the action in Eq.(6) satisfies the analogous (twistor) $\leftrightarrow$ (phase space) equivalence for all the cases on our list.

3 General twistor / phase space transform

In this section we discuss a unified formula that explicitly gives the substitutes for the Penrose relations of Eq.(4). The formula was derived in [11][8][12] through 2T-physics techniques. In this paper we simply use it, demonstrate that it works, and explain the underlying fundamental reasons for its structure. Thus, the general twistor / phase space transform for *spinless particles* at the classical level is given by $Z^A = \binom{\mu^{\dot{\alpha}}}{\lambda_\alpha}$, and

$$\mu^{\dot{\alpha}} = -i\frac{X^{\dot{\alpha}\beta}}{X^{+'}}\lambda_\beta\,, \quad \lambda_\alpha\bar{\lambda}_{\dot{\beta}} = \left(X^{+'}P_{\alpha\dot{\beta}} - P^{+'}X_{\alpha\dot{\beta}}\right), \tag{12}$$

where

$$X^{\dot{\alpha}\beta} = \frac{1}{\sqrt{2}}X^\mu\left(\bar{\sigma}_\mu\right)^{\dot{\alpha}\beta} = \frac{1}{\sqrt{2}}\begin{pmatrix} -X^0 + X^3 & X_1 - iX_2 \\ X_1 + iX_2 & -X^0 - X^3 \end{pmatrix}, \tag{13}$$

$$P_{\alpha\dot{\beta}} \equiv \frac{1}{\sqrt{2}}P^\mu\left(\sigma_\mu\right)_{\alpha\dot{\beta}} = \frac{1}{\sqrt{2}}\begin{pmatrix} P^0 + P^3 & P_1 - iP_2 \\ P_1 + iP_2 & P^0 - P^3 \end{pmatrix}. \tag{14}$$

The $\left(X^M, P^M\right)$ are the SO$(4,2)$ vectors of 2T-physics in Eq.(3), labelled by $M = \pm', \mu$ or $M = 0', 1', \mu$, and $\mu = \pm, 1, 2$ or $\mu = 0, 1, 2, 3$. They satisfy the fundamental 2T-physics Sp$(2, R)$ constraints

$$X \cdot X = P \cdot P = X \cdot P = 0. \tag{15}$$

These are the three generators of the gauge symmetry $\mathrm{Sp}(2,R)$ and their vanishing implies that the physical phase space is the subspace that is $\mathrm{Sp}(2,R)$ gauge invariant.

The dot product is constructed with the $\mathrm{SO}(4,2)$ metric η^{MN}. Evidently the $\mathrm{SO}(4,2)$ generators $L^{MN}=X^MP^N-X^NP^M$ commute with the three $\mathrm{Sp}(2,R)$ generators X^2, P^2, $X\cdot P$ since the latter are constructed as dot products. The metric η^{MN} is conveniently taken in a lightcone basis $X^M=\left(X^{+'},X^{-'},X^\mu\right)$ in the extra $(1,1)$ dimensions, $X^{\pm'}=\frac{1}{\sqrt{2}}\left(X^{0'}\pm X^{1'}\right)$, and $\mu=\pm,1,2$ labels the Minkowski subspace also in a lightcone basis with $X^\pm=\frac{1}{\sqrt{2}}\left(X^0\pm X^3\right)$

$$\begin{aligned}
ds^2 &= dX^MdX^N\eta_{MN}=-2dX^{+'}dX^{-'}+dX^\mu dX^\nu\eta_{\mu\nu} &(16)\\
&= -\left(dX^{0'}\right)^2+\left(dX^{1'}\right)^2-\left(dX^0\right)^2+\left(dX^1\right)^2+\left(dX^2\right)^2+\left(dX^3\right)^2 &(17)\\
&= -2dX^{+'}dX^{-'}-2dX^+dX^-+\left(dX_1\right)^2+\left(dX_2\right)^2. &(18)
\end{aligned}$$

The three properties that $Z^A=\binom{\mu^{\dot\alpha}}{\lambda_\alpha}$ must satisfy are:

1. First the helicity constraint $\bar Z_AZ^A=0$ for spinless particles is trivially guaranteed by the phase space / twistor transform in Eq.(12)

$$\begin{aligned}
\bar Z_AZ^A &= \left(\bar\lambda_{\dot\alpha}\ \bar\mu^\alpha\right)\begin{pmatrix}\mu^{\dot\alpha}\\ \lambda_\alpha\end{pmatrix}=\bar\lambda_{\dot\alpha}\mu^{\dot\alpha}+\bar\mu^\alpha\lambda_\alpha &(19)\\
&= -i\bar\lambda_{\dot\alpha}\left(\frac{X^{\dot\alpha\beta}}{X^{+'}}\right)\lambda_\beta+i\bar\lambda_{\dot\alpha}\left(\frac{X^{\dot\alpha\beta}}{X^{+'}}\right)\lambda_\beta=0. &(20)
\end{aligned}$$

2. Second, the canonical structure of Eq.(9) takes the form

$$\begin{aligned}
S_0 &= i\int d\tau\bar Z_A\partial_\tau Z^A=i\int d\tau\left[\bar\lambda_{\dot\alpha}\partial_\tau\mu^{\dot\alpha}+\bar\mu^\alpha\partial_\tau\lambda_\alpha\right]\\
&= \int d\tau\left[\bar\lambda_{\dot\alpha}\partial_\tau\left(\frac{X^{\dot\alpha\beta}}{X^{+'}}\lambda_\beta\right)-\bar\lambda_{\dot\alpha}\frac{X^{\dot\alpha\beta}}{X^{+'}}\partial_\tau\lambda_\beta\right] &(21)\\
&= \int d\tau\partial_\tau\left(\frac{X^{\dot\alpha\beta}}{X^{+'}}\right)\lambda_\beta\bar\lambda_{\dot\alpha}=\int d\tau\ \partial_\tau\left(\frac{X^\mu}{X^{+'}}\right)\left(X^{+'}P_\mu-P^{+'}X_\mu\right) &(22)
\end{aligned}$$

   and we must show that this form reduces to the canonical structure $S_0=\int d\tau\left(\dot x\cdot p\right)$ for each of the cases in our list.

3. Third, the mass shell condition follows from the form $\lambda_\alpha\bar\lambda_{\dot\beta}=X^{+'}P_{\alpha\dot\beta}-P^{+'}X_{\alpha\dot\beta}$. The left hand side is a rank one matrix with zero determinant and positive trace as seen from Eq.(10), and this requires that the right hand side must satisfy

$$\left(P^\mu-\frac{P^{+'}}{X^{+'}}X^\mu\right)^2=0,\ \left(X^{+'}P^0-P^{+'}X^0\right)>0. \qquad (23)$$

   We must show that these imply the mass shell conditions for all the cases in our list. These properties will be demonstrated in sections (4-8) below.

It is worth to point out that our twistor formula in Eq.(12) is related to a deeper structure. The expressions in Eq.(12) are equivalent to, and were derived from, the following more insightful $\mathrm{Sp}(2,R)$ gauge invariant expressions [12]

$$\mu^{\dot{\alpha}}=\begin{pmatrix} L^{+'-'}+L^{+\ -}+iL^{12} \\ -\sqrt{2}(L^{+1}+iL^{+2}) \end{pmatrix}\frac{e^{i\phi}}{\sqrt{4L^{+'+}}},\ \lambda_{\alpha}=\begin{pmatrix} 2L^{+'+} \\ \sqrt{2}(L^{+'1}+iL^{+'2}) \end{pmatrix}\frac{ie^{i\phi}}{\sqrt{4L^{+'+}}}, \tag{24}$$

Here the L^{MN} are the generators of $\mathrm{SO}(4,2)$

$$L^{MN}=X^{M}P^{N}-X^{N}P^{M}. \tag{25}$$

We emphasize that L^{MN} are $\mathrm{Sp}(2,R)$ gauge invariant. Inserting these L^{MN} into Eq.(24), and using the $\mathrm{Sp}(2,R)$ gauge singlet constraints of Eq.(15), gives the general twistor formula in Eq.(12). Furthermore, we note that, up to an overall factor, the twistor $Z^{A}=\binom{\mu}{\lambda}$ in the form of Eq.(24) is just the first column of the 4×4 matrix

$$\frac{1}{4}L_{MN}\Gamma^{MN}=\frac{1}{2}\begin{pmatrix} L^{+'-'}+\frac{1}{2}L_{\mu\nu}\bar{\sigma}^{\mu\nu} & -i\sqrt{2}L^{-'\mu}\ \bar{\sigma}_{\mu} \\ i\sqrt{2}L^{+'\mu}\ \sigma_{\mu} & -L^{+'-'}+\frac{1}{2}L_{\mu\nu}\sigma^{\mu\nu} \end{pmatrix} \tag{26}$$

which appears in Eq.(1), and is written in an appropriate gamma matrix basis[3] for the $\Gamma^{M},\bar{\Gamma}^{M}$. The other three columns of this matrix give other equivalent forms of the twistor Z^{A} up to different overall factors. Except for an undetermined phase, the different overall factors are fixed (as in 24) by requiring that Eq.(1) is satisfied. By inserting the explicit $L^{MN}=X^{[M}P^{N]}$ and using Eq.(1) we have

$$\frac{1}{4i}L_{MN}\left(\Gamma^{MN}\right)^{A}{}_{B} = \frac{1}{2i}\left[(X\cdot\Gamma)\left(P\cdot\bar{\Gamma}\right)-(P\cdot\Gamma)\left(X\cdot\bar{\Gamma}\right)\right]^{A}{}_{B} \tag{27}$$

$$= Z^{A}\bar{Z}_{B}-\frac{1}{4}\delta^{A}{}_{B}\ \left(Z_{C}\bar{Z}^{C}\right). \tag{28}$$

From this form one can see that at the classical level Z satisfies the zero eigenvalue equations $\left(X\cdot\bar{\Gamma}\right)Z=\left(P\cdot\bar{\Gamma}\right)Z=0$ since Z is proportional to any column of the matrix (27) and X^{M},P^{M} are subject to the constraints in Eq.(15). These zero eigenvalue conditions are equivalent to the generalized Penrose type transform in Eqs.(12,24), and can be further generalized to any dimension [12] directly in the form of Eq.(27,28).

Moreover, the structure of these formulas guarantee that the inverse relation between the L^{MN} and Z^{A} given in Eq.(1) also holds automatically. Hence the formulas in Eq.(12) or Eq.(24) give the general twistor transform for spinless particles. They

[3] The $\mathrm{SO}(4,2)=\mathrm{SU}(2,2)$ gamma matrices Γ^{M} in the 4×4 basis, or the $\bar{\Gamma}^{M}$ in the $\bar{4}\times\bar{4}$ basis, are taken as follows $\Gamma^{\pm'}=i\sqrt{2}\tau^{\pm}\times 1$, $\Gamma^{i}=\tau_{3}\times\sigma^{i}$, $\Gamma^{0}=-1\times 1$, while $\bar{\Gamma}^{M}$ are the same as the Γ^{M} for $M=\pm',i$, but for $M=0$ we have $\bar{\Gamma}^{0}=-\Gamma^{0}=1\times 1$. With this definition we have $\Gamma^{M}\bar{\Gamma}^{N}+\Gamma^{N}\bar{\Gamma}^{M}=2\eta^{MN}$ and $\Gamma^{MN}=\frac{1}{2}\left(\Gamma^{M}\bar{\Gamma}^{N}-\Gamma^{N}\bar{\Gamma}^{M}\right)$. Then $\frac{1}{2}\Gamma_{MN}L^{MN}=-\Gamma^{+'-'}L^{+'-'}+\ \frac{1}{2}L_{\mu\nu}\Gamma^{\mu\nu}-\Gamma^{+'}_{\mu}L^{-'\mu}-\Gamma^{-'}_{\mu}L^{+'\mu}$ takes the form given in Eq.(26). This choice of gamma matrices is consistent with the $\mathrm{SU}(2,2)$ metric $\eta=\tau_{1}\times 1$ used to define $\bar{Z}=Z^{\dagger}\eta=\left(\bar{\lambda}\ \ \bar{\mu}\right)$ for the fundamental quartet $Z=\binom{\mu}{\lambda}$. According to this metric we should have $\eta\left(Z\bar{Z}\right)^{\dagger}\eta = Z\bar{Z}$. Then Eq.(1) requires $\eta\left(\Gamma^{MN}\right)^{\dagger}\eta=-\Gamma^{MN}$, and this follows from the property $\eta\left(\Gamma^{M}\right)^{\dagger}\eta=-\bar{\Gamma}^{M}$ satisfied by our choice of gamma matrices.

apply not only to massless particles but to all the other cases in our list. The generalization of these expressions to spinning particles will be discussed elsewhere.

Now let us comment on some general properties of $\mathrm{Sp}(2,R)$ gauge fixing that will be applied to our formula in order to generate the twistors for the systems in our list. The $\left(X^M, P^M\right)$ transform as a doublet under $\mathrm{Sp}(2,R)$ for every M, as can be seen by commuting the $\left(X^M, P^M\right)$ with $\mathrm{Sp}(2,R)$ generators in Eq.(15). The L^{MN} are $\mathrm{Sp}(2,R)$ gauge invariant since they commute with those generators. Therefore the physical space consists of arbitrary functions $f\left(L^{MN}\right)$ [20]. Since these are all $\mathrm{Sp}(2,R)$ gauge invariant, their $\mathrm{SO}(4,2)$ properties cannot change by making gauge choices under $\mathrm{Sp}(2,R)$. Therefore every holographic picture of the $(4+2)$ phase space that is obtained by making $\mathrm{Sp}(2,R)$ gauge choices must reproduce the same $\mathrm{SO}(4,2)$ representation. Hence the physical states and operators can be classified as traceless tensors $T_{M_1\cdots M_n, N_1\cdots N_n}(L)$ of $\mathrm{SO}(4,2)$ constructed from powers of the antisymmetric L^{MN}. These correspond to the double-row Young tableaux $\begin{smallmatrix}\square\square\square\square\cdots\square\square_n\\ \square\square\square\square\cdots\square\square_n\end{smallmatrix}$ times scalar functions of the Casimirs[4] of $\mathrm{SO}(4,2)$. These tensors can be used, in any $\mathrm{Sp}(2,R)$ gauge fixed form to classify the same set of physical states or operators[5].

There could also be physical states or operators $f\left(L^{MN}\right)$ that cannot be expanded in powers of L^{MN}. In particular, as we see above, we can also construct the spinor representation of $\mathrm{SO}(4,2)$ from the L^{MN}, namely the twistor $Z^A\left(L^{MN}\right)$ in Eq.(24). This implies that we may construct physical states from the twistors. Our generalized twistors Z^A in Eq.(24) are functions of the L^{MN}, up to the gauge dependent overall phase $e^{i\phi}$ which could change under the $\mathrm{Sp}(2,R)$ gauge transformations. As already noted in the previous section, the overall phase of the twistor is indeed $\mathrm{U}(1)$ gauge dependent and unphysical. However the $\mathrm{U}(1)$ singlet condition applied on physical states as explained in footnote (1) requires only homogeneous functions of Z, so the physical space is equivalent whether written in terms of twistors or in terms of the L^{MN}. This key observation is the underlying reason for the same twistor Z^A to be equivalent to the various phase spaces that are derived by gauge fixing $\mathrm{Sp}(2,R)$.

So, by taking previously obtained solutions of gauge fixing and solving the constraints of 2T-physics [20][21][24], and plugging the resulting gauge fixed forms of $\left(X^M, P^M\right)$ into the gauge invariant formula in Eq.(24), or its equivalent in Eq.(12), we will obtain explicit phase space expressions for our twistor formula. There remains to check that these expressions explicitly have the properties analogous to those given in Eqs.(7,9,11) as formulated in Eqs.(22,23). Of course, these properties are already guaranteed by the symmetries and structures we have outlined above, but it will be useful and revealing to demonstrate them explicitly for each case in our list.

Let us first test the general twistor formula for the massless particle gauge in 2T-physics. In this fixed gauge we have $X^{+'}=1$ and $P^{+'}=0$, and two of the three

[4] All the $\mathrm{SO}(d,2)$ Casimir eigenvalues in our system $L^{MN}=X^{[M}P^{N]}$ vanish in the physical sector at the classical level due to the constraints in Eq.(15). However, they are non-zero, but fixed numbers, at the quantum level after taking into account quantum ordering. Thus $C_2\left(SO\left(d,2\right)\right)=1-d^2/2$, which corresponds to the singleton representation, as first explained in [20][25]. This is an $\mathrm{Sp}(2,R)$ gauge invariant result. Hence, the twistors for all the cases in our list provide another form of the singleton representation for $\mathrm{SO}(4,2)$.

[5] For a particular application in a 2T-physics gauge useful for classifying high spin fields, see [26].

constraints $X^2 = X\cdot P = 0$ are solved explicitly as follows

$$X^M = \left(\overset{+'}{1},\ \overset{-'}{x^2/2},\ \overset{\mu}{x^\mu}\right), \tag{29}$$

$$P^M = (\ 0\ ,\ x\cdot p\ ,\ \ p^\mu)\,. \tag{30}$$

The third $\mathrm{Sp}(2,R)$ gauge choice has not been made in order keep Lorentz covariance, hence there remains the constraint $P^2 = -2P^{+'}P^{-'} + P^\mu P_\mu = p^2 = 0$ to be imposed, which is the third $\mathrm{Sp}(2,R)$ generator. Then the various cross products give the $L^{MN} = X^M P^N - X^N P^M$ in the form

$$L^{+'-'} = x\cdot p,\ L^{+'\mu} = p^\mu,\ L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu,\ L^{-'\mu} = \frac{x^2}{2}p^\mu - x^\mu x.p, \tag{31}$$

which are recognized as the generators of $\mathrm{SO}(4,2)$ conformal transformations of the $3+1$ dimensional phase space at the classical level. Inserting these expressions into our general twistor equivalence formula in Eq.(24) or Eq.(12), we derive the Penrose relations of Eq.(4), which already satisfy the desired properties in Eqs.(22,23).

Furthermore, we can easily check the inverse relation, that the gauge fixed form of μ,λ in Eq.(4) reproduce the L^{MN} through the general formula of Eq.(1)

$$\left(Z\bar{Z}\right)^A_{\ B} = \begin{pmatrix} \mu^{\dot\alpha}\bar\lambda_{\dot\beta} & \mu^{\dot\alpha}\bar\mu^\beta \\ \lambda_\alpha\bar\lambda_{\dot\beta} & \lambda_\alpha\bar\mu^\beta \end{pmatrix} = \begin{pmatrix} -i\bar{x}p & \bar{x}p\bar{x} \\ p & ip\bar{x} \end{pmatrix} \tag{32}$$

$$= \frac{1}{2i}\begin{pmatrix} L^{+'-'}+\frac{1}{2}L^{\mu\nu}\bar\sigma_{\mu\nu} & -i\sqrt{2}L^{-'\mu}\ \bar\sigma_\mu \\ i\sqrt{2}L^{+'\mu}\ \sigma_\mu & -L^{+'-'}+\frac{1}{2}L^{\mu\nu}\sigma_{\mu\nu} \end{pmatrix}. \tag{33}$$

Here the L^{MN} in the last matrix is computed in each block from the twistors as

$$\mu\bar\lambda = -i\bar{x}\lambda\bar\lambda = -i\bar{x}p = \frac{1}{2i}\left(L^{+'-'}+\frac{1}{2}L^{\mu\nu}\bar\sigma_{\mu\nu}\right),\ \text{etc.}, \tag{34}$$

with $L^{+'-'} = x^\mu p_\mu$, $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, and so on. We see then that the L^{MN} computed from the twistors are precisely the $L^{MN} = X^M P^N - X^N P^M$ of Eq.(31) computed from the gauge fixed vectors X^M, P^M Eq.(29,30), in agreement with our general statement in Eq.(1). This test case, combined with the reasoning provided above, imply that we will succeed just in the same way with the other 2T-physics gauge choices that correspond to the list in the introduction. So, each $\mathrm{Sp}(2,R)$ gauge choice in 2T-physics will automatically generate the corresponding twistors through our general formula in Eq.(12), or through the equivalent $\mathrm{Sp}(2,R)$ gauge invariant formula Eq.(24).

4 Twistors for massive relativistic particles

We will discuss the massive particle in two versions which correspond to different looking gauge fixed forms of the X^M, P^M. These are of course related to each other by

Sp$(2,R)$ gauge transformations, but nevertheless seem independently interesting from the point of view of 1T-physics. An Sp$(2,R)$ gauge choice that describes the massive particle was given in the second paper in [21], and is described in the Appendix. The zero mass limit of this gauge does not smoothly connect to the massless particle gauge in Eqs.(29,30) and seems to display singularities. This is not a problem since the connection should be only up to a Sp$(2,R)$ gauge transformations. It is possible to perform Sp$(2,R)$ gauge transformations to obtain a second form of the massive particle gauge which has a smooth zero mass limit. We discuss the smooth case in this section and the non-smooth case in the Appendix.

The smooth massive particle gauge is given by the following gauge choice of the X^M, P^M components

$$X^M = \left(\overset{+'}{\frac{1+a}{2a}}, \ \overset{-'}{\frac{x^2 a}{1+a}}, \ \overset{\mu}{x^\mu} \right) \tag{35}$$

$$P^M = \left(\frac{-m^2}{2(x\cdot p)a}, \ (x\cdot p)\,a\ , \ p^\mu \right) \tag{36}$$

where

$$a \equiv \sqrt{1+\frac{m^2x^2}{(x\cdot p)^2}}. \tag{37}$$

Note that $(x\cdot p)a$ is nonsingular as $x\cdot p\to 0$. To arrive to the above form two gauge choices have been made and the two constraints $X\cdot X = X\cdot P$ have been solved, thus eliminating 4 functions. This fixes completely the four components $X^{\pm'}, P^{\pm'}$ in terms of the remaining independent phase space degrees of freedom $x^\mu(\tau), p^\mu(\tau)$. Note that $x^\mu(\tau), p^\mu(\tau)$ are dynamical variables while the constant mass m emerges as a "modulus" from a gauge fixed version of the other phase space variables $X^{\pm'}, P^{\pm'}$. The third constraints $P\cdot P = 0$ gives the mass shell condition for the massive particle

$$0 = P\cdot P = -2P^{+'}P^{-'} + P^\mu P_\mu = p^2 + m^2. \tag{38}$$

With this parametrization, the 2T action in Eq.(3) reduces to the action of the relativistic massive particle

$$S = \int d\tau \ \left(\dot{X}^M P^N - \frac{1}{2}A^{ij}X_i^M X_j^N \right) \eta_{MN} = \int d\tau \left(\dot{x}^\mu p_\mu - \frac{1}{2}A^{22}\left(p^2+m^2\right) \right). \tag{39}$$

and this justifies the parametrization given in Eqs.(35,36).

In the limit $m\to 0$, as $a\to 1$ this gauge smoothly reduces to the massless particle gauge discussed in the previous section. We should warn the reader that the phase space degrees of freedom (x^μ, p^μ) of the massive particle are not the same as those of the massless particle. By applying an Sp$(2,R)$ *local* transformation, the massive doublets in Eqs.(35,36) can be transformed to the massless doublets in Eqs.(29,30). The Sp$(2,R)$ gauge transformation may be regarded as a canonical transformation

that includes the time components (and hence changes the Hamiltonian of the massive particle to the Hamiltonian of the massless particle).

The $SO(4,2)$ generators $L^{MN}=X^{[M}P^{N]}$ take the following form in this gauge

$$
\begin{aligned}
L^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, \qquad L^{+'-'}=(x\cdot p)\,a, && (40)\\
L^{+'\mu} &= \frac{1+a}{2a}p^\mu + \frac{m^2}{2\,(x\cdot p)\,a}x^\mu && (41)\\
L^{-'\mu} &= \frac{x^2 a}{1+a}p^\mu - (x\cdot p)\,a x^\mu && (42)
\end{aligned}
$$

For general m these L^{MN} close under Poisson brackets to form the Lie algebra of $SO(4,2)$ at the classical level. In the massless limit, $m\to 0$, $a\to 1$, these L^{MN} reduce to the familiar expressions for conformal transformations as given in Eq.(31).

With the appropriate quantum ordering the Lie algebra must close at the quantum level, and must have the quadratic Casimir eigenvalue for the singleton representation $C_2=1-d^2/4$, which is $C_2=-3$ for $d=4$, to be consistent with $SO(4,2)$ covariant quantization of the system given in [20].

We study the twistor transform for the massive particle in this gauge by inserting $X^{+'},P^{+'},X^\mu,P^\mu$ in Eqs.(35,36) into Eq.(12). This gives our new twistor transform for the massive particle

$$\mu^{\dot\alpha}=-ix^{\dot\alpha\beta}\lambda_\beta\frac{2a}{1+a},\quad \lambda_\alpha\bar\lambda_{\dot\beta}=\frac{1+a}{2a}p_{\alpha\dot\beta}+\frac{m^2}{2(x\cdot p)a}x_{\alpha\dot\beta}. \tag{43}$$

We know from Eq.(19) that this form satisfies automatically the $\bar ZZ=\bar\lambda\mu+\bar\mu\lambda=0$ constraint. We now turn to the canonical structure induced from twistors as formulated in Eq.(22) and compute it in the present gauge

$$
\begin{aligned}
S_0 &= i\int d\tau \bar Z_A\partial_\tau Z^A = i\int d\tau\left[\bar\lambda_{\dot\alpha}\partial_\tau\mu^{\dot\alpha}+\bar\mu^\alpha\partial_\tau\lambda_\alpha\right] && (44)\\
&= \int d\tau\frac{\partial}{\partial\tau}\left(\frac{x^\mu}{X^{+'}}\right)\left(X^{+'}p_\mu - P^{+'}x_\mu\right) && (45)\\
&= \int d\tau\left(\begin{array}{c}(\partial_\tau x)\cdot p-\frac{P^{+'}}{X^{+'}}(\partial_\tau x)\cdot x\\ -\partial_\tau\ln\left(X^{+'}\right)\left(x\cdot p-\frac{P^{+'}}{X^{+'}}x\cdot x\right)\end{array}\right) && (46)\\
&= \int d\tau\left[\dot x\cdot p+\partial_\tau\left(\frac{-m^2x^2}{(x\cdot p)\,(1+a)}\right)\right]. && (47)
\end{aligned}
$$

The total derivative can be dropped, so $S_0=\int d\tau\,(\dot x\cdot p)$ gives the correct canonical structure in phase space. Hence the canonical twistor space is equivalent to the canonical phase space.

Finally we investigate the mass shell condition that is induced by the twistors as given in Eq.(23). Inserting $X^{+'}=\frac{1+a}{2a}$ and $P^{+'}=\frac{-m^2}{2(x\cdot p)a}$ we compute Eq.(23). In the

present case this takes the form

$$0 = \left(p^\mu - \frac{P^{+'}}{X^{+'}}x^\mu\right)^2 = \left(p^\mu + \frac{m^2}{(x\cdot p)\,(1+a)}x^\mu\right)^2 = \left(p^2+m^2\right) \tag{48}$$

$$0 < \left(X^{+'}p^0 - P^{+'}x^0\right) = \frac{1+a}{2a}p^0 + \frac{m^2}{2(x\cdot p)a}x^0 \tag{49}$$

This shows that the twistors in Eq.(43) induce the correct mass shell condition for a relativistic massive particle. Since $a>0$ the positivity condition becomes

$$\frac{m^2}{x\cdot p}x^0 > -\,(1+a)\,p^0. \tag{50}$$

In this equation we write $x\cdot p = -x^0p^0 + \mathbf{x}\cdot\mathbf{p}$ and $x^2 = -\left(x^0\right)^2 + \mathbf{x}^2$, insert the on-shell value $p^0 = \pm\sqrt{\mathbf{p}^2+m^2}$, and then solve for the allowed region for x^0. First consider the limit of large values of $|x^0|$ for which $a\left(x^0\right)$ has a leading term independent of x^0 and is approximated by $a\sim\frac{|\mathbf{p}|}{\sqrt{\mathbf{p}^2+m^2}}$. Then x^0 drops out and the inequality becomes $(m^2/p^0) < (1+\frac{|\mathbf{p}|}{\sqrt{\mathbf{p}^2+m^2}})p^0$. This is satisfied only for positive p^0, hence the on-shell solution for p^0 is

$$p^0>0,\ p^0=\sqrt{\mathbf{p}^2+m^2}. \tag{51}$$

Inserting this into the inequality we search for the allowed regions for x^0, and find that all values of x^0 are permitted.

Therefore, the twistor representation given in Eq.(43) describes correctly a massive particle of positive energy[6].

We can now check that the inverse relation also holds, namely that the L^{MN} computed from the twistors are precisely the $L^{MN} = X^MP^N - X^NP^M$ of Eq.(31) computed from the vectors X^M, P^M. Thus, analogous to Eq.(32) we now compute by using the twistor transform in Eq.(43)

$$\left(Z\bar{Z}\right)^A{}_B = \begin{pmatrix} \mu^{\dot\alpha}\bar\lambda_{\dot\beta} & \mu^{\dot\alpha}\bar\mu^{\beta} \\ \lambda_\alpha\bar\lambda_{\dot\beta} & \lambda_\alpha\bar\mu^\beta \end{pmatrix} \tag{52}$$

$$= \begin{pmatrix} -i\bar{x}\left(p+\frac{m^2}{(x\cdot p)(1+a)}x\right) & \bar{x}\left(p+\frac{m^2}{(x\cdot p)(1+a)}x\right)\bar{x}\frac{2a}{1+a} \\ \frac{1+a}{2a}p+\frac{m^2}{2(x\cdot p)a}x & i\left(p+\frac{m^2}{(x\cdot p)(1+a)}x\right)\bar{x} \end{pmatrix} \tag{53}$$

$$= -i\begin{pmatrix} L^{+'-'}+\frac{1}{2}L^{\mu\nu}\bar\sigma_{\mu\nu} & -i\sqrt{2}L^{-'\mu}\,\bar\sigma_\mu \\ i\sqrt{2}L^{+'\mu}\,\sigma_\mu & -L^{+'-'}+\frac{1}{2}L^{\mu\nu}\sigma_{\mu\nu} \end{pmatrix} \tag{54}$$

where the second line is obtained by manipulations such as

$$\mu\bar\lambda = -i\frac{2a}{1+a}\bar{x}\lambda\bar\lambda = -i\frac{2a}{1+a}\bar{x}\left(\frac{1+a}{2a}p+\frac{m^2}{2(x\cdot p)a}x\right) = -i\left(L^{+'-'}+\frac{1}{2}L^{\mu\nu}\bar\sigma_{\mu\nu}\right),\ \text{etc.,}$$

[6] If a were taken as the negative square root in Eq.(37), then we would conclude that p^0 must also be the negative square root.

The $L^{MN} = X^M P^N - X^N P^M$ obtained through the twistors in this way are identical to those given in Eqs.(40-42) which were computed from the vectors X^M, P^M, in agreement with the general relation in Eq.(1).

Hence we have successfully constructed the twistor transform for the massive particle. We have shown that there is an SO$(4,2)$ =SU$(2,2)$ symmetry that is non-linearly realized in phase space (x^μ, p^μ), but is linearly realized and is an explicit symmetry of the action in the twistor version Eq.(6) or 2T-physics version Eq.(3). The SU$(2,2)$ symmetry is identical for the massive or the massless particle, and in both cases corresponds to the unitary singleton representation of SO$(4,2)$ whose Casimir eigenvalues are independent of the mass parameter (e.g. $C_2 = 0$ at the classical level, but $C_2 = -3$ at the quantum level). However, the same SO$(4,2)$ is realized in rather different forms in the phase spaces of massive versus massless particles. This unfamiliar result, which was discovered in 2T-physics, has now taken a new facade through twistor space.

5 Twistors for nonrelativistic particle

The nonrelativistic particle is given by the following gauge choice of the X^M, P^M components in $(4+2)$ dimensional phase space in 2T-physics (see second paper in [21])

$$
\begin{aligned}
X^M &= \Big(\overset{+'}{t}, \overset{-'}{\frac{\mathbf{r}\cdot\mathbf{p} - tH}{m}}, \overset{0}{u}, \overset{i=1,2,3}{\mathbf{r}^i} \Big), &(55)\\
P^M &= \left(m, \ H \ , \ 0 \ , \ \mathbf{p}^i \right) &(56)
\end{aligned}
$$

where u is fixed by

$$
u^2 \equiv \mathbf{r}^2 - \frac{2t}{m}\mathbf{r}\cdot\mathbf{p} + \frac{2t^2}{m}H. \tag{57}
$$

To arrive to the above form two gauge choices have been made: $P^0(\tau) = 0$, and $P^{+'}(\tau) = m$ a constant, for all τ. Solving the two constraints $X\cdot X = X\cdot P = 0$ fixes completely $X^{-'}$ and X^0 in terms of the phase space variables $(t(\tau), \mathbf{r}^i(\tau))$ and $(H(\tau), \mathbf{p}^i(\tau))$ as given above. The third constraint becomes $P\cdot P = -2mH + \mathbf{p}^2 = 0$, which implies that $H = \mathbf{p}^2/2m$ is the Hamiltonian for the non-relativistic particle. With this parametrization, the 2T action in Eq.(3) reduces to the action of the nonrelativistic massive particle

$$
\begin{aligned}
S &= \int d\tau \left(\dot{X}^M P^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN} &(58)\\
&= \int d\tau \left(-\dot{t}H + \dot{\mathbf{r}}\cdot\mathbf{p} - \frac{1}{2} A^{22}\left(-2mH + \mathbf{p}^2\right) \right). &(59)
\end{aligned}
$$

This justifies the parametrization given in Eq.(55,56). If the remaining gauge freedom is fixed as $t(\tau) = \tau$, and the constraint $0 = P\cdot P = -2mH + \mathbf{p}^2$ is imposed explicitly,

this action becomes the standard action of the non-relativistic particle

$$S=\int d\tau\left(\dot{\mathbf{r}}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}\right). \tag{60}$$

In this case $X^{-'},X^0$ become $X^{-'}=\mathbf{p}\cdot\mathbf{r}-\boldsymbol{\tau}\frac{\mathbf{p}^2}{2m}$ and $X^0=u=\pm\left|\mathbf{r}-\boldsymbol{\tau}\frac{\mathbf{p}}{m}\right|$ respectively.

Twistors may be discussed either in the fully gauge fixed form or in the partially gauge fixed form, but it is preferable not to choose the last gauge and treat (t,H) as canonical variables, while applying the constraint $H-\frac{\mathbf{p}^2}{2m}=0$ on physical states.

The SO$(4,2)$ generators $L^{MN}=X^{[M}P^{N]}$ are

$$\begin{aligned} L^{ij} &= \mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i,\ \ L^{+'-'}=2tH-\mathbf{r}\cdot\mathbf{p},\, L^{+'i}=t\mathbf{p}^i-m\mathbf{r}^i \qquad (61)\\ L^{+'0} &= -mu,\ \ L^{-'0}=-Hu,\ \ L^{-'i}=\frac{\mathbf{r}\cdot\mathbf{p}}{m}\mathbf{p}^i-H\left(\mathbf{r}^i+\frac{t}{m}\mathbf{p}^i\right), \qquad (62)\end{aligned}$$

One can check that by using the Poisson brackets $\{t,H\}=-1,\ \{\mathbf{r}^i,\mathbf{p}^j\}=\delta^{ij}$ the Lie algebra of SO$(4,2)$ is satisfied at the classical level. Alternatively, in the gauge $t(\tau)=\tau$ and $H=\frac{\mathbf{p}^2}{2m}$, we use only $\{\mathbf{r}^i,\mathbf{p}^j\}=\delta^{ij}$ and treat τ as a parameter, to show that the Lie algebra of SO$(4,2)$ is satisfied at the classical level at any τ. It is harder to establish the Lie algebra at the quantum level due to the non-linear ordering problems presented by the square root in $u=\pm\left(\mathbf{r}^2-\frac{2t}{m}\mathbf{r}\cdot\mathbf{p}+\frac{2t^2}{m}H\right)^{1/2}$.

We construct the new twistor transform for the nonrelativistic particle by inserting $X^{+'},P^{+'},X^\mu,P^\mu$ in Eqs.(55,56) into Eq.(12). This gives

$$\mu^{\dot\alpha}=-i\frac{1}{t}\left(-u+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\sigma}}\right)^{\dot\alpha\beta}\lambda_\beta,\ \ \lambda_\alpha\bar\lambda_{\dot\beta}=\left(mu+(t\tilde{\mathbf{p}}-m\tilde{\mathbf{r}})\cdot\tilde{\boldsymbol{\sigma}}\right)_{\alpha\dot\beta}. \tag{63}$$

We know from Eq.(19) that the $\bar ZZ=\bar\lambda\mu+\bar\mu\lambda=0$ constraint for spinless particles is automatically satisfied. We now turn to the canonical structure induced from twistors as formulated in Eq.(22) and compute it in the present gauge

$$\begin{aligned} S_0 &= i\int d\tau\bar Z_A\partial_\tau Z^A=i\int d\tau\left[\bar\lambda_{\dot\alpha}\partial_\tau\mu^{\dot\alpha}+\bar\mu^\alpha\partial_\tau\lambda_\alpha\right] \qquad (64)\\ &= \int d\tau\frac{\partial}{\partial\tau}\left(\frac{X^\mu}{X^{+'}}\right)\left(X^{+'}P_\mu-P^{+'}X_\mu\right) \qquad (65)\\ &= \int d\tau\left(\partial_\tau\left(\frac{u}{t}\right)mu+\partial_\tau\left(\frac{1}{t}\tilde{\mathbf{r}}\right)\cdot(t\tilde{\mathbf{p}}-m\tilde{\mathbf{r}})\right) \qquad (66)\\ &= \int d\tau\left[-\dot tH+\dot{\mathbf{r}}\cdot\mathbf{p}+\partial_\tau\left(\frac{mu^2-m\mathbf{r}^2}{2t}\right)\right]. \qquad (67)\end{aligned}$$

The total derivative can be dropped, so S_0 gives the correct canonical structure in phase space.

Next we investigate the mass shell condition that is induced by the twistors as given in Eq.(23). In the present case this takes the form

$$
\begin{aligned}
0 &= \left(P^{\mu}-\frac{P^{+'}}{X^{+'}}X^{\mu}\right)^2=-\left(\frac{m}{t}u\right)^2+\left(\mathbf{p}-\frac{m}{t}\mathbf{r}\right)^2, &(68)\\
0 &< \left(X^{+'}P^{0}-P^{+'}X^{0}\right)=-mu &(69)
\end{aligned}
$$

From this we see that we must choose the negative square root for the u^2 given in Eq.(57)

$$
u=-\left(\mathbf{r}^2-\frac{2t}{m}\mathbf{r}\cdot\mathbf{p}+\frac{2t^2}{m}H\right)^{1/2} \tag{70}
$$

Inserting this form we find

$$
0=-\left(\frac{m}{t}u\right)^2+\left(\mathbf{p}-\frac{m}{t}\mathbf{r}\right)^2=-2mH+\mathbf{p}^2 \tag{71}
$$

This shows that the correct mass shell condition is induced by the twistors in Eq.(63). After using $H=\mathbf{p}^2/2m$ we find $u=-\left|\mathbf{r}-t\frac{\mathbf{p}}{m}\right|$, which shows there is no condition on the range of t for the square root to be real. Therefore, the twistor representation given in Eq.(63) describes correctly a nonrelativistic particle.

Next we investigate the inverse relation analogous to Eq.(32). Using the twistor transform in Eq.(63) we have

$$
\left(Z\bar{Z}\right)^{A}{}_{B}=\begin{pmatrix}\mu^{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}} & \mu^{\dot{\alpha}}\bar{\mu}^{\beta}\\ \lambda_{\alpha}\bar{\lambda}_{\dot{\beta}} & \lambda_{\alpha}\bar{\mu}^{\beta}\end{pmatrix}=\frac{1}{2i}\begin{pmatrix}L^{+'-'}+\frac{1}{2}L^{\mu\nu}\bar{\sigma}_{\mu\nu} & -i\sqrt{2}L^{-'\mu}\,\bar{\sigma}_{\mu}\\ i\sqrt{2}L^{+'\mu}\,\sigma_{\mu} & -L^{+'-'}+\frac{1}{2}L^{\mu\nu}\sigma_{\mu\nu}\end{pmatrix} \tag{72}
$$

where

$$
\begin{aligned}
\mu\bar{\lambda} &= -i\frac{1}{t}\left(-u+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\sigma}}\right)\left(mu+\left(t\tilde{\mathbf{p}}-m\tilde{\mathbf{r}}\right)\cdot\tilde{\boldsymbol{\sigma}}\right), &(73)\\
\mu\bar{\mu} &= \frac{1}{t^2}\left(-u+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\sigma}}\right)\left(mu+\left(t\tilde{\mathbf{p}}-m\tilde{\mathbf{r}}\right)\cdot\tilde{\boldsymbol{\sigma}}\right)\left(-u+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\sigma}}\right), &(74)\\
\lambda\bar{\lambda} &= \left(mu+\left(t\tilde{\mathbf{p}}-m\tilde{\mathbf{r}}\right)\cdot\tilde{\boldsymbol{\sigma}}\right), &(75)\\
\lambda\bar{\mu} &= i\frac{1}{t}\left(mu+\left(t\tilde{\mathbf{p}}-m\tilde{\mathbf{r}}\right)\cdot\tilde{\boldsymbol{\sigma}}\right)\left(-u+\tilde{\mathbf{r}}\cdot\tilde{\boldsymbol{\sigma}}\right). &(76)
\end{aligned}
$$

The $L^{MN}=X^{M}P^{N}-X^{N}P^{M}$ obtained through the twistors by comparing the matrices in Eq.(72) are identical to those given in Eqs.(61-62) which were computed directly from the vectors X^M,P^M. This is in agreement with the general relation in Eq.(1).

Hence we have successfully constructed the twistor transform for the massive particle. This shows that the SU$(2,2)$ symmetry, that is linearly realized on the twistors Z^A or on the vectors $\left(X^M,P^M\right)$, is a non-linearly realized hidden symmetry of the action $S=\int d\tau\left(\dot{\mathbf{r}}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}\right)$ of the non-relativistic particle, which indeed is the case (see second paper in [21]). Furthermore, there exists a duality among the nonrelativistic particle, the relativistic massive or massless particles as well as all the other particle systems discussed in this paper, since they are all constructed from the same twistor Z^A, or the same 2T phase space X^M,P^M, and they all are realizations of the singleton representation of SO$(4,2)$.

6 Twistors for particles in $\mathrm{AdS}_{4-n}\times\mathrm{S}^n$

We consider a particle moving in $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ spaces for $n = 0, 1, 2, \cdots, (d-2)$. In general, to describe these spaces we make the following $\mathrm{Sp}(2,R)$ gauge choice in $(d+2)$ dimensional phase space in 2T-physics (see second paper in [21])

$$\begin{aligned} X^M &= \frac{R}{|\mathbf{y}|}\left(\overset{+'}{R}\ ,\quad \overset{-'}{\frac{x^2+y^2}{2R}},\qquad \overset{m}{x^m}\quad ,\quad \overset{I}{\mathbf{y}^I}\right),\quad \begin{array}{c} m=0,1,\cdots,(d-2-n) \\ I=1,\cdots,(n+1) \end{array} && (77) \\ P^M &= \frac{|\mathbf{y}|}{R}\left(\ 0\ ,\ \frac{1}{R}(x\cdot p+y\cdot k),\quad p^m\quad ,\quad \mathbf{k}^I\ \right). && (78) \end{aligned}$$

To arrive at this form two $\mathrm{Sp}(2,R)$ gauge choices have been made and two $\mathrm{Sp}(2,R)$ constraints $X\cdot X = X\cdot P = 0$ have been solved explicitly, and the solution is parameterized by $\left(x^m, \mathbf{y}^I\right)$. One of the $\mathrm{Sp}(2,R)$ gauge choices is $P^{+'}(\tau) = 0$ for all τ, and the second gauge choice is $\sqrt{X^I(\tau)\,X^I(\tau)} = R$ with a τ independent R. Note that R is a modulus that arises from the degrees of freedom in the larger phase space of 2T-physics. The $n+1$ coordinates $X^I(\tau) = R\frac{\mathbf{y}^I}{|\mathbf{y}|}$ which consist of a unit vector times the constant radius R, represent motion on the sphere S^n embedded in an $n+1$ dimensional volume. The constraint $0 = X\cdot X = \left(X^I\right)^2 + \tilde{X}^2$ implies that the $(d-n+1)$ coordinates $\tilde{X} = \left(X^{+'}, X^{-'}, X^m\right)$, which have signature $((d-n-1), 2)$, satisfy $\tilde{X}\cdot\tilde{X} = -R^2$. The solution of the constraint $\tilde{X}\cdot\tilde{X} = -R^2$, which automatically represents motion on the space AdS_{d-n}, is parameterized by the $d-n$ coordinates $(x^m\ , |\mathbf{y}|)$ as given above.

For $d = 4$ we have a total of four dimensions $\left(x^m(\tau), y^I(\tau)\right)$ that remain after gauge fixing. In this gauge the flat $(4,2)$ metric generates the $\mathrm{AdS}_{4-n}\times\mathrm{S}^n$ metric,

$$\begin{aligned} ds^2 &= dX^M dX^N \eta_{MN} = \frac{R^2}{y^2}\left[(dx^m)^2 + \left(d\mathbf{y}^I\right)^2\right] && (79) \\ &= \frac{R^2}{y^2}\left[(dx^m)^2 + (dy)^2\right] + R^2\,(d\mathbf{\Omega})^2\,, && (80) \end{aligned}$$

where we have decomposed $\mathbf{y}^I$ in spherical coordinates $\mathbf{y}^I = y\mathbf{\Omega}^I$ into its radial $y = |\mathbf{y}|$ and angular $\mathbf{\Omega}^I = \frac{\mathbf{y}^I}{|\mathbf{y}|}$ parts, and wrote $(d\mathbf{y})^2 = (dy)^2 + y^2\,(d\mathbf{\Omega})^2$. Evidently, $\frac{R^2}{y^2}\left[(dx^m)^2 + (dy)^2\right]$ is the AdS_{4-n} metric and $R^2\,(d\mathbf{\Omega})^2$ is the S^n metric. The same parametrization can be used in $d+2$ phase space for any d to construct $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ [21][23].

In the case of AdS_4 we take $n = 0$ which corresponds to the following coordinates

$$X^M = \left(\overset{+'}{\frac{R^2}{y}}, \quad \overset{-'}{\frac{x \cdot x + y^2}{2y}}, \quad \overset{m=0,1,2}{\frac{Rx^m}{y}}, \quad R \right) \tag{81}$$

$$P^M = \left(\quad 0 \ , \ \frac{y}{R^2}(x{\cdot}p + yk), \ \frac{p^m y}{R}, \quad \frac{ky}{R} \right). \tag{82}$$

where $X^3(\tau) = R$, is the gauge choice and $X^m(\tau) = \frac{Rx^m(\tau)}{y(\tau)}$, $m = 0, 1, 2$. The dot products with $x^m(\tau), p^m(\tau)$ involves the 3-dimensional Minkowski metric η^{mn}. The structure of X^M, P^M is such that the 2T-physics action reduces to the action of a particle moving on AdS$_4$

$$S = \int d\tau \left(\dot{X}^M P^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN} \tag{83}$$

$$= \int d\tau \left(\dot{x}^m p_m + \dot{y} k - \frac{1}{2} A^{22} \left(p^m p_m + k^2 \right) \frac{y^2}{R^2} \right). \tag{84}$$

This justifies the parametrization given in Eqs.(77,78).

The last factor multiplying A^{22} is the Laplacian of AdS$_d$ at the classical level, namely $p_\mu p_\nu g^{\mu\nu} = \left(p^m p_m + k^2\right) \frac{y^2}{R^2}$, where $g^{\mu\nu}$ is the inverse of the metric in Eq.(80) for $n = 0$. When it is quantum ordered into a Hermitian form $\left[y \left(p^m p_m + k^2\right) y\right]$ that is applied on physical states ψ, namely $y \left(\partial^m \partial_m + \partial_y^2\right) y \psi \left(x^m, y\right) = 0$, this form gives the correct Laplacian for a particle on AdS$_d$ after a re-scaling $\psi = (-g)^{1/4} \phi = \left(\frac{R}{y}\right)^{d/2} \phi$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) + \frac{d(d-2)}{4R^2} \phi = 0. \tag{85}$$

The quantum ordering introduces a quantized mass term $m_\phi^2 = d(d-2)/4R^2$ which is required by the SO$(d, 2)$ invariance of the AdS$_d$ particle at the quantum level, or by the corresponding field theory, as shown in ([21]). Note that the SO$(d, 2)$ symmetry (SO$(4, 2)$ in the case of AdS$_4$) is larger than the commonly mentioned SO$(d-1, 2)$ symmetry of AdS$_d$ space (SO$(3, 2)$ in the case of AdS$_4$). The extra symmetry (which is noticeable from the left side of Eq.(80)) is hidden in the (x_m, y) basis and its presence was first noticed through the 2T-physics formulation [21]. This symmetry becomes evident also in the twistor basis below.

Inserting the coordinates $X^\mu = \frac{R}{y}(x^m, y)$, $P^\mu = \frac{y}{R}(p^m, k)$ and $X^{+'} = \frac{R^2}{y}$, $P^{+'} = 0$ in the twistor transform of Eq.(12) we obtain the twistors for the AdS$_4$ particle,

$$\mu^{\dot{\alpha}} = -i \frac{X^{\dot{\alpha}\beta}}{X^{+'}} \lambda_\beta = -i \frac{y}{\sqrt{2} R^2} \left(\frac{R}{y} x^m \bar{\sigma}_m + R \sigma_3 \right)^{\dot{\alpha}\beta} \lambda_\beta \tag{86}$$

$$= -i \frac{1}{\sqrt{2} R} \left(x^m \bar{\sigma}_m + y \sigma_3 \right)^{\dot{\alpha}\beta} \lambda_\beta, \tag{87}$$

and

$$
\begin{aligned}
\lambda_\alpha \bar{\lambda}_{\dot{\beta}} &= X^{+'} P_{\alpha\dot{\beta}} = \frac{R^2}{\sqrt{2}y} \left(\frac{y}{R} p^m \sigma_m + \frac{y}{R} k \sigma_3 \right) && (88) \\
&= \frac{R}{\sqrt{2}} \left(p^m \sigma_m + k \sigma_3 \right). && (89)
\end{aligned}
$$

Thus, in this gauge the twistor structure is very similar to the massless particle in $d = 4$, with $x^\mu = (x^m, y)$ and $p^\mu = (p^m, k)$, while the form of the twistor $Z = \binom{\mu}{\lambda}$ displays the $SU(2,2)$ $=SO(4,2)$ symmetry in the fundamental representation.

Let's examine the canonical structure of phase space induced by the canonical structure of the twistors, as described in Eq.(22)

$$
\begin{aligned}
S_0 &= i \int d\tau \bar{Z}_A \partial_\tau Z^A = i \int d\tau \left[\bar{\lambda}_{\dot{\alpha}} \partial_\tau \mu^{\dot{\alpha}} + \bar{\mu}^\alpha \partial_\tau \lambda_\alpha \right] && (90) \\
&= \int d\tau \frac{\partial}{\partial \tau} \left(\frac{X_\mu}{X^{+'}} \right) \left(X^{+'} P_\mu \right) && (91) \\
&= \int d\tau \frac{\partial}{\partial \tau} \left(\frac{x^\mu}{R^2} \right) (R p_\mu) = \int d\tau \left(\dot{x}^m p_m + \dot{y} k \right) && (92)
\end{aligned}
$$

So, the canonical structure is correct. Next we examine the on-shell condition

$$
0 = \left(P^\mu - \frac{P^{+'}}{X^{+'}} X^\mu \right)^2 = P^2 = \frac{y^2}{R^2} \left(p^m p_m + k^2 \right) = 0. \tag{93}
$$

This gives the correct Laplacian for the massless conformal particle in the curved AdS_4 space as explained after Eq.(84). The positivity condition trivially gives $p^0 > 0$.

Next we examine the inverse relation

$$
\left(Z \bar{Z} \right)^A{}_B = \begin{pmatrix} \mu^{\dot{\alpha}} \bar{\lambda}_{\dot{\beta}} & \mu^{\dot{\alpha}} \bar{\mu}^\beta \\ \lambda_\alpha \bar{\lambda}_{\dot{\beta}} & \lambda_\alpha \bar{\mu}^\beta \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} L^{+'-'} + \frac{1}{2} L^{\mu\nu} \bar{\sigma}_{\mu\nu} & -i\sqrt{2} L^{-'\mu} \, \bar{\sigma}_\mu \\ i\sqrt{2} L^{+'\mu} \, \sigma_\mu & -L^{+'-'} + \frac{1}{2} L^{\mu\nu} \sigma_{\mu\nu} \end{pmatrix} \tag{94}
$$

It can be verified that the twistors correctly reproduce the L^{MN} constructed from the vectors in Eq.(81,82), as shown in Eqs.(97-101) below, and that they correctly close into the $SO(4,2)$ Lie algebra at the classical level. Hence we have constructed the correct twistor description of the AdS_4 particle.

The twistor version explicitly shows that the AdS_4 particle has the $SO(4,2)$ symmetry which is larger than the commonly expected $SO(3,2)$ symmetry. The larger symmetry is also evident by writing the metric in the form of Eq.(80) and noting the symmetries of the left hand side $ds^2 = dX^M dX^N \eta_{MN}$. The reason for the extra symmetry is in the fact that R is a gauge fixed form of an extra coordinate as in Eq.(81), and this is not noticed in common approaches in discussing AdS symmetries. In any case our AdS_4 twistor transform makes it evident that the system has the $SU(2,2)$ symmetry linearly realized in the twistor version in Eq.(6) or in the 2T-physics version in Eq.(3).

We now turn to the cases of $\mathrm{AdS}_{4-n}\times\mathrm{S}^n$ for $n=1,2$ and compute the canonical structure induced by the twistors in Eq.(22), $S_0=\int d\tau\frac{\partial}{\partial\tau}\left(\frac{|\mathbf{y}|X^\mu}{R^2}\right)\left(\frac{R^2}{|\mathbf{y}|}P_\mu\right)$. We write $x^\mu=\left(x^m,y^I\right)$, $p_\mu=(p_m,k_I)$ with $I=1,\cdots,n{+}1$, and use the definitions $X^\mu=\frac{R}{|\mathbf{y}|}x^\mu$, $P_\mu=p_\mu\frac{|\mathbf{y}|}{R}$ that follows from Eqs.(77,78). Then we obtain the correct canonical structure

$$S_0=\int d\tau\frac{\partial}{\partial\tau}\left(\frac{|\mathbf{y}|\,X^\mu}{R^2}\right)\left(\frac{R^2}{|\mathbf{y}|}P_\mu\right)=\int d\tau\frac{\partial}{\partial\tau}\left(\frac{x^\mu}{R}\right)(Rp_\mu)=\int d\tau\left(\dot{x}^m p_m+\dot{\mathbf{y}}^I\mathbf{k}^I\right).$$

Similarly, the mass-shell condition in Eq.(23) gives

$$\begin{aligned} 0 &= \left(P^\mu-\frac{P^{+'}}{X^{+'}}X^\mu\right)^2=P^2=\frac{y^2}{R^2}\left(p^m p_m+\mathbf{k}^2\right) && (95)\\ &= \frac{y^2}{R^2}\left(p^m p_m+k^2\right)+\frac{1}{2}L^{IJ}L_{IJ} && (96)\end{aligned}$$

where in the last line we rewrote the vector dot product $\mathbf{k}^2$ in terms of the radial and angular momenta. The Casimir operator $\frac{1}{2}L^{IJ}L_{IJ}$ for $\mathrm{SO}(n+1)$ is the correct Laplacian on S^n. After quantum ordering into a Hermitian operator $y\left(p^m p_m+k^2\right)y+\frac{1}{2}L^{IJ}L_{IJ}=0$ applied on physical states $\psi\left(x^m,y,\mathbf{\Omega}\right)=(-g)^{1/4}\phi$, this gives the correct Laplacian for the massless particle in the curved $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ space for the metric of Eq.(80), with an induced mass $m_\phi^2=(d-2n)(d-2)/4R^2$ at the quantum level, as discussed in [21]. Note that the mass vanishes for the case of $d=2n$, which applies to $\mathrm{AdS}_2\times\mathrm{S}^2$ in the present case. The positivity condition $\left(X^{+'}P^0-P^{+'}X^0\right)>0$ gives trivially $p^0>0$.

Similarly to the AdS_4 case above the inverse relations give the $\mathrm{SO}(4,2)$ generators consistently in terms of twistors or in terms of the gauge fixed vectors X^M,P^M. These are given by for every $n=0,1,2$ as

$$\begin{aligned} L^{IJ} &= \mathbf{y}^I\mathbf{k}^J-\mathbf{y}^J\mathbf{k}^I,\quad L^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu, && (97)\\ L^{+'-'} &= \mathbf{y}\cdot\mathbf{k}+x\cdot p,\quad L^{+'\mu}=p^\mu,\quad L^{+'I}=\mathbf{k}^I, && (98)\\ L^{-'\mu} &= \frac{1}{2}\left(x^2+\mathbf{y}^2\right)p^\mu-\left(x\cdot p+\mathbf{y}\cdot\mathbf{k}\right)x^\mu, && (99)\\ L^{-'I} &= \frac{1}{2}\left(x^2+\mathbf{y}^2\right)\mathbf{k}^I-\left(x\cdot p+\mathbf{y}\cdot\mathbf{k}\right)\mathbf{y}^I, && (100)\\ L^{\mu I} &= x^\mu\,\mathbf{k}^I-p^\mu\mathbf{y}^I. && (101)\end{aligned}$$

Thus the spaces $\mathrm{AdS}_{4-n}\times\mathrm{S}^n$ have hidden symmetries $\mathrm{SO}(4,2)$ which is larger than the commonly discussed $\mathrm{SO}(3-n,2)\times\mathrm{SO}(n+1)$. Again the crucial observation in understanding what is missed in common discussions, is that R is a gauge fixed form of an additional coordinate that corresponds to the AdS×S radius $\sqrt{X^I(\tau)\,X^I(\tau)}=R$. The full $\mathrm{SO}(4,2)$ acts non-linearly on the remaining degrees of freedom after the gauge is fixed, but it acts linearly before the gauge fixing. The twistor transform makes the hidden symmetry evident in the twistor version of the system. The larger

hidden symmetry in phase space version of $\mathrm{AdS}_{d-n}\times\mathrm{S}^n$ is discussed in detail at the quantum level for any d in the second paper[7] in [21].

7 Twistors for particle on $R\times\mathbf{S}^3$

We make two $\mathrm{Sp}(2,R)$ gauge choices and solve two constraints to obtain the following gauge fixed form of the $(4+2)$ dimensional phase space in 2T-physics

$$
\begin{aligned}
X^M &= R\left(\overset{0'}{\cos t}\,,\ \ \overset{0}{\sin t},\ \ \ \overset{I=1,2,3,4}{\hat{\mathbf{r}}^I}\right), &(102)\\
P^M &= \frac{1}{R}\left(-H\sin t,\ H\cos t\ ,\ \hat{\mathbf{r}}_J\mathbf{L}^{JI}\right), &(103)
\end{aligned}
$$

where $\hat{\mathbf{r}}$ is the unit vector $\hat{\mathbf{r}}\,(\tau) = \mathbf{r}^I\,(\tau)\,/\,|\mathbf{r}|$ that defines the motion on the sphere S^3 embedded in four Euclidean dimensions, and $\left(X^I X^I\right)^{1/2} = |\mathbf{r}\,(\tau)| = R$ is the radial coordinate in spherical coordinates that has been gauge fixed to be a constant for all τ. Evidently $\hat{\mathbf{r}}^I$ can be parameterized in terms of three angles, but we will not need to give an explicit parametrization.

Defining the SO(4) rotation generators $L^{IJ} = \mathbf{r}^I\mathbf{p}^J - \mathbf{r}^J\mathbf{p}^I$, we note that $\frac{1}{R}\hat{\mathbf{r}}_J\mathbf{L}^{JI} = \mathbf{p}^I - \hat{\mathbf{r}}\hat{\mathbf{r}}\cdot\mathbf{p}$ is purely SO(4) angular momentum since the radial momentum $\hat{\mathbf{r}}\cdot\mathbf{p}$ has been subtracted. In fact $\hat{\mathbf{r}}\cdot\mathbf{p}$ drops out everywhere, which is equivalent to choosing the second gauge as $\hat{\mathbf{r}}\cdot\mathbf{p}=0$ for all τ. Then the $\mathrm{Sp}(2,R)$ constraints $X^2 = X\cdot P = 0$ are explicitly solved with the above form. In this gauge the 2T action in Eq.(3) reduces to

$$
\begin{aligned}
S &= \int d\tau\ \left(\dot{X}^M P^N - \frac{1}{2}A^{ij}X_i^M X_j^N\right)\eta_{MN} &(104)\\
&= \int d\tau\left(-H\partial_\tau t + \mathbf{L}^{JI}\hat{\mathbf{r}}_J\partial_\tau\hat{\mathbf{r}}_I - \frac{1}{2R^2}A^{22}\left(-H^2 + \frac{1}{2}L^{IJ}L_{IJ}\right)\right) &(105)
\end{aligned}
$$

where we have used $\left(\hat{\mathbf{r}}_J\mathbf{L}^{JI}\right)^2 = \frac{1}{2}L^{IJ}L_{IJ}$. The first two terms define the canonical structure that gives the quantum rules for the particle on $R\times S^3$, namely $[t,H] = -i$, $[\mathbf{L}_{IJ},\hat{\mathbf{r}}_K] = i\hat{\mathbf{r}}_J\delta_{IK} - i\hat{\mathbf{r}}_I\delta_{JK}$ and $[\mathbf{L}_{IJ},\mathbf{L}_{KL}]$ =SO(4) Lie algebra. The remaining constraint

$$
P\cdot P = \frac{1}{R^2}\left(-H^2 + \left(\hat{\mathbf{r}}_J\mathbf{L}^{JI}\right)^2\right) \tag{106}
$$

appears as the coefficient of the gauge field A^{22}. We could fix the remaining gauge symmetry by fixing the gauge $t\,(\tau) = \tau$, and solving the constraint explicitly $H^2 = \frac{1}{2}L^{IJ}L_{IJ}$. But we will work more generally without choosing this gauge and impose the constraint on physical states. Then the physical states are the symmetric traceless

[7] To adapt the quantum results of [21] we must use a translation of notations. Instead of the canonical variables $(\mathbf{y},\mathbf{k})$ that we used here, ref.[21] uses the canonical set $(\mathbf{u},\tilde{\mathbf{k}})$ (but $\tilde{\mathbf{k}}$ is called $\mathbf{k}$ in [21]). The relation between these parameterizations at the quantum level is given by $\mathbf{y} = \mathbf{u}/\mathbf{u}^2$ and $\mathbf{k} = \tilde{\mathbf{k}}\mathbf{u}^2 - \left(\tilde{\mathbf{k}}\cdot\mathbf{u} + \mathbf{u}\cdot\tilde{\mathbf{k}}\right)\mathbf{u}$ as described in Eqs.(74–79) and footnote 4 in ref.[21].

SO(4) tensors $T_{I_1\cdots I_l}(\hat{\mathbf{r}})$, for which H^2 takes the values of the Casimir operator $H^2 = \frac{1}{2}L^{IJ}L_{IJ} = l(l+2)$, $l = 0,1,2,\cdots$ of the SO(4) rotations on S^3. So, in this gauge the 2T-physics system is interpreted as a particle moving on $R\times\mathrm{S}^3$.

We now construct the twistors for this system by applying the general formulas in Eq.(12)

$$\mu^{\dot{\alpha}} = -i\frac{X^{\dot{\alpha}\beta}}{X^{+'}}\lambda_\beta\,,\quad \lambda_\alpha\bar{\lambda}_{\dot{\beta}} = \left(X^{+'}P_{\alpha\dot{\beta}} - P^{+'}X_{\alpha\dot{\beta}}\right) \tag{107}$$

where as usual $X^{\dot{\alpha}\beta} = \frac{1}{\sqrt{2}}X^\mu\left(\bar{\sigma}_\mu\right)^{\dot{\alpha}\beta}$ and $P_{\alpha\dot{\beta}} \equiv \frac{1}{\sqrt{2}}P^\mu\left(\sigma_\mu\right)_{\alpha\dot{\beta}}$, and following the definitions in Eqs.(102,103) we identify $X^{+'}, P^{+'}, X^\mu, P^\mu$ as follows

$$\begin{aligned} X^{+'} &= \frac{R}{\sqrt{2}}\left(\cos t + \hat{\mathbf{r}}_4\right), \quad X^\mu = R\begin{pmatrix} \overset{0}{\sin t} & \overset{i=1,2,3}{\hat{\mathbf{r}}^i}\end{pmatrix},\ \left(\hat{\mathbf{r}}_i\right)^2 + \left(\hat{\mathbf{r}}_4\right)^2 = 1 \qquad (108)\\ P^{+'} &= \frac{1}{\sqrt{2}R}\left(-H\sin t + \hat{\mathbf{r}}^j\mathbf{L}_{j4}\right),\ P^\mu = \frac{1}{R}\begin{pmatrix} \overset{0}{H\cos t} & \overset{i=1,2,3}{\hat{\mathbf{r}}^j\mathbf{L}_{ji}}\end{pmatrix} \qquad (109)\end{aligned}$$

We could parameterize $\hat{\mathbf{r}}_4 = \cos\theta$, $\hat{\mathbf{r}}^i = n^i\sin\theta$, where n^i is a unit vector in 3 dimensions, but we will not need such an explicit parametrization. With the above definitions we find the twistor transform for the particle on S^3 as follows

$$\begin{aligned} \mu^{\dot{\alpha}} &= -i\frac{\sqrt{2}\left(-\sin t + \hat{\mathbf{r}}^i\sigma^i\right)^{\dot{\alpha}\beta}}{\left(\cos t + \hat{\mathbf{r}}_4\right)}\lambda_\beta, \qquad (110)\\ \lambda_\alpha\bar{\lambda}_{\dot{\beta}} &= \frac{1}{\sqrt{2}}\begin{bmatrix} \left(\cos t + \hat{\mathbf{r}}_4\right)\left(H\cos t + \hat{\mathbf{r}}^j\mathbf{L}_{ji}\sigma^i\right) \\ -\left(-H\sin t + \hat{\mathbf{r}}^j\mathbf{L}_{j4}\right)\left(\sin t + \hat{\mathbf{r}}^i\sigma^i\right)\end{bmatrix}_{\alpha\dot{\beta}} \qquad (111)\end{aligned}$$

Next we check the canonical structure

$$\begin{aligned} S_0 &= i\int d\tau\bar{Z}_A\partial_\tau Z^A = i\int d\tau\left[\bar{\lambda}_{\dot{\alpha}}\partial_\tau\mu^{\dot{\alpha}} + \bar{\mu}^\alpha\partial_\tau\lambda_\alpha\right] \qquad (112)\\ &= \int d\tau Tr\left\{\partial_\tau\left(\frac{X^\mu}{X^{+'}}\right)\left(X^{+'}P_\mu - P^{+'}X_\mu\right)\right\} \qquad (113)\\ &= \int d\tau\left(-H\partial_\tau t + \mathbf{L}^{JI}\hat{\mathbf{r}}_J\partial_\tau\hat{\mathbf{r}}_I\right) \qquad (114)\end{aligned}$$

This is the correct canonical structure for $R\times S^3$, as described following Eq.(105).

Turning to the on-shell condition we find det $\left(\lambda\bar{\lambda}\right) = 0$ implies

$$\begin{aligned} 0 &= \left(P_\mu - \frac{P^{+'}}{X^{+'}}X_\mu\right)^2\\ &= -\frac{1}{R^2}\left(H\cos t - \frac{-H\sin t + \hat{\mathbf{r}}^j\mathbf{L}_{j4}}{\cos t + \hat{\mathbf{r}}_4}\sin t\right)^2 + \frac{1}{R^2}\left(\hat{\mathbf{r}}^j\mathbf{L}_{ji} - \frac{-H\sin t + \hat{\mathbf{r}}^j\mathbf{L}_{j4}}{\cos t + \hat{\mathbf{r}}_4}\hat{\mathbf{r}}^i\right)^2\\ &= \frac{1}{R^2}\left(-H^2 + \frac{1}{2}L^{IJ}L_{IJ}\right), \qquad (115)\end{aligned}$$

which imposes the correct constraint $H^2 = \frac{1}{2}L^{IJ}L_{IJ}$.

Finally, the sign condition is

$$\begin{aligned}
\left[(\cos t + \hat{\mathbf{r}}_4)\, H \cos t + \left(H \sin t - \hat{\mathbf{r}}^j \mathbf{L}_{j4}\right) \sin t\right] \quad &> \quad 0 &(116)\\
\text{or } \left[H + H\hat{\mathbf{r}}_4 \cos t - \hat{\mathbf{r}}^j \mathbf{L}_{j4} \sin t\right] \quad &> \quad 0 &(117)
\end{aligned}$$

This is satisfied for all t. Therefore the twistors correctly describe the particle on $R{\times}\mathrm{S}^3$.

8 Twistors for H-atom

For the H-atom gauge let us consider the following gauge choice [21] for the 4+2 dimensional phase space

$$\begin{aligned}
X^M \quad &= \quad F\left(\overset{0'}{\cos u}\ ,\ \overset{1'}{-\frac{1}{\alpha}\sqrt{-2H}\mathbf{r}\cdot\mathbf{p}},\ \overset{0}{\sin u}\ ,\ \overset{i}{(\frac{1}{r}\mathbf{r}^i - \frac{\mathbf{r}\cdot\mathbf{p}}{\alpha}\mathbf{p}^i)}\right) &(118)\\
P^M \quad &= \quad G\left(-\sin u,\quad (1-\frac{r\mathbf{p}^2}{\alpha})\ ,\ \cos u\ ,\quad \sqrt{-2H}\frac{r}{\alpha}\mathbf{p}^i\right) &(119)\\
GF \quad &= \quad \frac{\alpha}{\sqrt{-2H}},\qquad u = \frac{\sqrt{-2H}}{\alpha}\left(\mathbf{r}\cdot\mathbf{p} - 2\tau H\right),\quad H = \frac{\mathbf{p}^2}{2} - \frac{\alpha}{r}, &(120)
\end{aligned}$$

where we have fixed the three gauge degrees of freedom and imposed all three constraints $X^2 = P^2 = X \cdot P = 0$. Using these coordinates the 2T action in Eq.(3) reduces to

$$\begin{aligned}
S \quad &= \quad \int d\tau\ \left(\dot{X}^M P^N - \frac{1}{2}A^{ij}X_i^M X_j^N\right)\eta_{MN} &(121)\\
&= \quad \int d\tau\ \left(\mathbf{p}^i \partial_\tau \mathbf{r}_i - H\right). &(122)
\end{aligned}$$

This is the non-relativistic H-atom action in three space dimensions. Now that we have introduced the 2T-physics gauge, we are going to consider the twistor version for this case. As in the previous cases, we construct the twistors for this system by applying the general formulas in Eq.(12)

$$\mu^{\dot{\alpha}} = -i\frac{X^{\dot{\alpha}\beta}}{X^{+'}}\lambda_\beta\,,\quad \lambda_\alpha\bar{\lambda}_{\dot{\beta}} = \left(X^{+'}P_{\alpha\dot{\beta}} - P^{+'}X_{\alpha\dot{\beta}}\right) \qquad (123)$$

where the coordinates $X^{+'}, P^{+'},\ X^\mu, P^\mu$ come from (118,119) as

$$\begin{aligned}
X^{+'} \quad &= \quad \frac{F}{\sqrt{2}}\left(\cos u - \frac{1}{\alpha}\sqrt{-2H}\mathbf{r}\cdot\mathbf{p}\right),\quad X^\mu = F\left(\overset{0}{\sin u}\ \ \overset{i=1,2,3}{\frac{1}{r}\mathbf{r}^i - \frac{\mathbf{r}\cdot\mathbf{p}}{\alpha}\mathbf{p}^i}\right), &(124)\\
P^{+'} \quad &= \quad \frac{G}{\sqrt{2}}\left(-\sin u + \ 1 - \frac{r\mathbf{p}^2}{\alpha}\right),\qquad P^\mu = G\left(\overset{0}{\cos u}\ \ \overset{i=1,2,3}{\sqrt{-2H}\frac{r}{\alpha}\mathbf{p}^i}\right). &(125)
\end{aligned}$$

So the twistor transform becomes

$$\mu^{\dot{\alpha}} = -i\frac{\left(-\sin u + \frac{1}{r}\mathbf{r}^i\boldsymbol{\sigma}^i - \frac{\mathbf{r}\cdot\mathbf{p}}{\alpha}\mathbf{p}^i\boldsymbol{\sigma}^i\right)^{\dot{\alpha}\beta}}{\cos u - \frac{1}{\alpha}\sqrt{-2H}\mathbf{r}\cdot\mathbf{p}}\lambda_\beta \tag{126}$$

$$\lambda_\alpha\bar{\lambda}_{\dot{\beta}} = \frac{\alpha}{2\sqrt{-2H}}\begin{pmatrix} \left(\cos u - \frac{1}{\alpha}\sqrt{-2H}\mathbf{r}\cdot\mathbf{p}\right)\left(\cos u + \sqrt{-2H}\frac{r}{\alpha}\mathbf{p}^i\boldsymbol{\sigma}^i\right) \\ -\left(-\sin u + \ 1 - \frac{r\mathbf{p}^2}{\alpha}\right)\left(\sin u + \frac{1}{r}\mathbf{r}^i\boldsymbol{\sigma}^i - \frac{\mathbf{r}\cdot\mathbf{p}}{\alpha}\mathbf{p}^i\boldsymbol{\sigma}^i\right)\end{pmatrix}_{\alpha\dot{\beta}} \tag{127}$$

If we substitute the twistor in this case in Eq. (22), we can see that the canonical structure reduces to the Lagrangian for the Hydrogen atom, plus a total derivative, $\partial_\tau\left(3\tau H - 2\mathbf{r}\cdot\mathbf{p}\right)$, that is dropped in the last line below

$$\begin{aligned} S_0 &= i\int d\tau \bar{Z}_A\partial_\tau Z^A = i\int d\tau\left[\bar{\lambda}_{\dot{\alpha}}\partial_\tau\mu^{\dot{\alpha}} + \bar{\mu}^\alpha\partial_\tau\lambda_\alpha\right] &(128)\\ &= \int d\tau\frac{\partial}{\partial\tau}\left(\frac{X^\mu}{X^{+'}}\right)\left(X^{+'}P_\mu - P^{+'}X_\mu\right) &(129)\\ &= \int d\tau\left[\partial_\tau X^\mu P_\mu - \frac{P^{+'}}{X^{+'}}\partial_\tau X^\mu X_\mu + \left(X^{+'}X^\mu P_\mu - P^{+'}X^\mu X_\mu\right)\partial_\tau\frac{1}{X^{+'}}\right] &(130)\\ &= \int d\tau\left(\dot{\mathbf{r}}\cdot\mathbf{p} - H + \partial_\tau\left(3\tau H - 2\mathbf{r}\cdot\mathbf{p}\right)\right) &(131)\\ &\to \int d\tau\left(\dot{\mathbf{r}}\cdot\mathbf{p} - H\right) &(132)\end{aligned}$$

Also, the mass shell condition $\left(P_\mu - X_\mu P^{+'}/X^{+'}\right)^2 = 0$ that is required by $\det\left(\lambda\bar{\lambda}\right) = 0$ is fulfilled for the twistors above when $H = \frac{\mathbf{p}^2}{2} - \frac{\alpha}{r}$.

Finally, the positivity condition $X^{+'}P_0 - P^{+'}X_0 > 0$ takes the form

$$\left[1 - \frac{\sqrt{-2H}}{\alpha}\mathbf{r}\cdot\mathbf{p}\cos u - \left(\ 1 - \frac{r\mathbf{p}^2}{\alpha}\right)\sin u\right] > 0. \tag{133}$$

This inequality can be written in terms of the dimensionless $v \equiv \frac{rp^2}{\alpha}$ with $v \leq 2$, and and the angle $\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} = \cos\theta$ as follows

$$\left[1 - \sqrt{2v - v^2}\cos\theta\cos u - (1 - v)\sin u\right] > 0 \tag{134}$$

This is satisfied for any $u(\tau)$, and therefore for any τ. Hence the twistors given above correctly describe the H-atom.

9 Conclusions and comments

We have constructed the twistors for an assortment of particle dynamical systems, including special examples of massless or massive particles, relativistic or non-relativistic, interacting or non-interacting, in flat space or curved spaces. More examples can be constructed in one to one correspondence with all other possible gauge choices that we

can make in 2T-physics. Our unified construction involves always the *same* twistor Z^A with only four complex degrees of freedom and subject to the *same* helicity constraint. Only the twistor to phase space transform differs from one case to another. Hence a unification of diverse particle dynamical systems is displayed by the fact that they all share the same twistor description.

Of course, this unification is equivalent to 2T-physics, except that in the present case it is expressed in terms of twistors instead of the 6 dimensional vectors X^M, P^M. Furthermore, the actions in the six dimensional phase space Eq.(3) or in the twistor space Eq.(6) are both $SO(4,2)$ $=SU(2,2)$ invariant and are physically equivalent. Either form of the action can be taken as the starting point to derive all of the results of this paper. The equivalence of the two actions is derived as two gauge fixed forms of the same 2T-physics action, one in the "particle gauge", and the other in the "twistor gauge", as explained in [11][7][8][12].

The twistor to phase space transform for the cases of $R\times S^3$ and H-atom seemed rather complicated. One of the reasons for this is that the natural evident symmetry for these cases is $SO(4)$, but the twistor components $Z^A = \binom{\mu^{\dot{\alpha}}}{\lambda_\alpha}$ are expressed in an $SL(2,C)$ $=SO(3,1)$ basis. The clash of the $SO(4)$ versus $SO(3,1)$ spinor bases makes the expressions for the twistor transform complicated. It is certainly possible to choose an $SO(4)$ $=SU(2)\times SU(2)$ basis to express the twistor components. This $SU(2)\times SU(2)$ is the natural compact subgroup of $SU(2,2)$ in the fundamental basis for which the metric takes the form of $\tau_3 \times 1$ instead of the $\tau_1 \times 1$ used in the $SO(3,1)$ basis (see footnote (3)). The twistor can then be expressed as $Z^A = \binom{a_i}{\bar{b}^I}$ and $\bar{Z}_A = (\bar{a}^i \ \ -b_I)$ where $i=1,2$ and $I=1,2$ refer to the doublets of the two different $SU(2)$'s. An overbar such as $\bar{b}^I$ implies hermitian conjugate of b_I. Inserting these into the kinetic term S_0 we learn from $\bar{Z}_A i\partial_\tau Z^A = i\bar{a}^i\partial_\tau a_i - ib_I\partial_\tau \bar{b}^I$ that the canonical structure is that of positive norm harmonic oscillators

$$\left[a_i, \bar{a}^j\right] = \delta_i^{\ j}, \quad \left[b_I, \bar{b}^J\right] = \delta_I^{\ J}. \tag{135}$$

Hence, twistor space corresponds to the unitary Fock space constructed from the oscillators. Equivalently one may use coherent states that diagonalize the oscillators. The twistor to phase space transforms in the $SO(4)$ basis can be given generally as

$$\begin{aligned} a_i &= -i\left(\frac{X^m\left(\bar{\sigma}_m\right)_{iJ}}{X^{0'}+iX^0}\right)\bar{b}^J, \quad \sigma^m = (\vec{\sigma}, i),\ \bar{\sigma}^m = (\vec{\sigma}, -i) = (\sigma^m)^\dagger, \\ a_i b_J &= \left(\bar{\sigma}^m\right)_{iJ}\left[\left(X_{0'}+iX_0\right)P_m - \left(P_{0'}+iP_0\right)X_m\right],\ m=1,2,3,4 \text{ for } SO(4). \end{aligned} \tag{136}$$

This should be compared to the $SL(2,C)$ basis in Eq.(12). Note that the factor $\frac{X^m\bar{\sigma}_m}{X^{0'}+iX^0}$ is a unitary matrix since $X^M X_M = -\left(X_{0'}\right)^2 - \left(X_0\right)^2 + \left(X_m\right)^2 = 0$. When applied to the cases of $R\times S^3$ and H-atom the expressions for the twistor transform are considerably simpler and more natural (we do not give the details, but this is straightforward). So it would be more desirable to use the $SO(4)$ basis if one attempts to use twistors for these or similar cases.

In this paper we concentrated on spinless particles in 4 dimensions. In future papers we will provide a similar construction of twistors for the corresponding spinning systems [19] and higher dimensions [27].

Acknowledgments

I. Bars was supported by the US Department of Energy under grant No. DE-FG03-84ER40168; M. Picón was supported by the Spanish Ministerio de Educación y Ciencia through the grant FIS2005-02761 and EU FEDER funds, the Generalitat Valenciana and by the EU network MRTN-CT-2004-005104 "Constituents, Fundamental Forces and Symmetries of the Universe". M. Picón wishes to thank the Spanish Ministerio de Educación y Ciencia for his FPU research grant, and the USC Department of Physics and Astronomy for kind hospitality.

10 Appendix (second version of massive particle)

A massive particle gauge is obtained by setting $P^{0'}(\tau) = 0$ and $P^{1'}(\tau) = m$ for all τ. Note that the mass m is identified as the component of momentum in a higher dimension. By solving explicitly the two constraints $X \cdot X = 0$ and $X \cdot P = 0$, the gauge fixed form of X^M, P^M become [21]

$$\begin{aligned} X^M &= \left(\overset{0'}{-\frac{x \cdot p}{m} a}, \ \overset{1'}{-\frac{x \cdot p}{m}}, \ \overset{\mu}{x^\mu} \right), \quad a^2 = 1 + \frac{m^2 x^2}{(x \cdot p)^2}, & (137) \\ P^M &= (\quad 0 \quad , \quad m \quad , \quad p^\mu) . & (138) \end{aligned}$$

The third constraint reduces to the mass shell condition for the massive particle $0 = P \cdot P = p^2 + m^2$. In this gauge the SO$(4,2)$ generators $L^{MN} = X^{[M} P^{N]}$ take the form

$$\begin{aligned} L^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, \quad L^{0'1'} = -(x \cdot p)\, a, & (139) \\ L^{0'\mu} &= -p^\mu \frac{x \cdot p}{m} a, \quad L^{1'\mu} = -\frac{x \cdot p}{m} p^\mu - m x^\mu & (140) \end{aligned}$$

Next we obtain the twistor relations for the massive particle by inserting $X^{+'}, P^{+'}, X^\mu, P^\mu$ that follow from Eqs.(137,138)

$$X^{+'} = -\frac{x \cdot p}{\sqrt{2} m} (1 + a), \quad P^{+'} = \frac{m}{\sqrt{2}}, \quad X^\mu = x^\mu, \ P^\mu = p^\mu \tag{141}$$

into Eq.(12). This gives

$$\begin{aligned} \mu^{\dot\alpha} &= i x^{\dot\alpha\beta} \lambda_\beta \frac{\sqrt{2} m}{x \cdot p} (1 + a)^{-1}, & (142) \\ \lambda_\alpha \bar\lambda_{\dot\beta} &= -p_{\alpha\dot\beta} \frac{x \cdot p}{\sqrt{2} m} (1 + a) - \frac{m}{\sqrt{2}} x_{\alpha\dot\beta}, & (143) \end{aligned}$$

We note the parallels as well as the differences compared to the massless case in Eq.(4). The zero mass limit does not seem to be smooth for X^M, P^M, but this is not a problem since this is only up to a Sp$(2,R)$ gauge transformations. In the second

massive particle gauge discussed in section (4) the zero mass limit for X^M, P^M is smooth.

We know from Eq.(19) that the $\bar{Z}Z = 0$ constraint is already satisfied. We now turn to the canonical structure as formulated in Eq.(22) and compute it in the present gauge

$$S_0 = \int d\tau \frac{\partial}{\partial\tau}\left(\frac{x^\mu}{X^{+'}}\right)\left(X^{+'}p_\mu - P^{+'}x_\mu\right) \tag{144}$$

$$= \int d\tau \left(\dot{x}\cdot p - \partial_\tau\left(x\cdot p\right)\right) \tag{145}$$

A little algebra shows that the extra total derivative term emerges when $X^{+'}, P^{+'}$ are inserted in the form

$$P^{+'}x\cdot\partial_\tau x + \left(X^{+'}\left(x\cdot p\right) - P^{+'}x^2\right)\frac{\partial_\tau X^{+'}}{\left(X^{+'}\right)^2} = \partial_\tau\left(x\cdot p\right). \tag{146}$$

The total derivative can be dropped, so $S_0 = \int d\tau\left(\dot{x}\cdot p\right)$, indeed gives the correct canonical structure.

Finally we check the mass shell condition as formulated in Eq.(23). This requires

$$\left(p^\mu - \frac{P^{+'}}{X^{+'}}x^\mu\right)^2 = 0,\ \left(X^{+'}p^0 - P^{+'}x^0\right) > 0 \tag{147}$$

Inserting $X^{+'}, P^{+'}$ we compute

$$0 = \left(p^\mu - \frac{P^{+'}}{X^{+'}}x^\mu\right)^2 = \frac{1}{2}\left(p^2 + m^2\right) \tag{148}$$

This is the massive particle mass shell condition.

Examining the positivity condition in Eq.(23), it becomes

$$m^2x^0 < -\left(1+a\right)\left(x\cdot p\right)p^0 \tag{149}$$

and using the mass shell condition that implies $p^0 = E = \pm\sqrt{\mathbf{p}^2 + m^2}$, we find

$$m^2x^0 < -E\left(-x^0E + \mathbf{x}\cdot\mathbf{p} + sign\left(a\right)\sqrt{\left(x^0E - \mathbf{x}\cdot\mathbf{p}\right)^2 + m^2\left(-x_0^2 + \mathbf{x}^2\right)}\right).$$

An analysis of this equation shows that the equality sign can be satisfied only for complex values of $x^0 = \frac{E}{\mathbf{p}^2}\left(\mathbf{x}\cdot\mathbf{p} \pm i\sqrt{\mathbf{x}^2\mathbf{p}^2 - \left(\mathbf{x}\cdot\mathbf{p}\right)^2}\right)$, while the inequality is satisfied for all values of x^0, for either sign of $E = \pm\sqrt{\mathbf{p}^2 + m^2}$, as well as for either sign of a. Hence, the twistors defined above correctly describe the massive particle.

References

[1] R. Penrose, "Twistor Algebra," J. Math. Phys. **8** (1967) 345; "Twistor theory, its aims and achievements, in Quantum Gravity", C.J. Isham et. al. (Eds.), Clarendon, Oxford 1975, p. 268-407; "The Nonlinear Graviton", Gen. Rel. Grav. **7** (1976) 171; "The Twistor Program," Rept. Math. Phys. **12** (1977) 65.

[2] R. Penrose and M.A. MacCallum, "An approach to the quantization of fields and space-time", Phys. Rept. **C6** (1972) 241; R. Penrose and W. Rindler, Spinors and space-time II, Cambridge Univ. Press (1986);

[3] E. Witten, "Perturbative gauge theory as a string theory in twistor space", Commun. Math. Phys. **252** (2004) 189 [arXiv:hep-th/0312171]; "Parity invariance for strings in twistor space", hep-th/0403199.

[4] N. Berkovits, "An Alternative string theory in twistor space for N=4 Super Yang-Mills", Phys. Rev. Lett. **93** (2004) 011601 [arXiv:hep-th/0402045].

[5] N. Berkovits and L. Motl, "Cubic twistorial string field theory", JHEP **0404** (2004) 56, [arXiv:hep-th/0403187].

[6] N. Berkovits and E. Witten, "Conformal supergravity in twistor-string theory", JHEP **0408** (2204) 009, [arXiv:hep-th/0406051].

[7] I. Bars, "Twistor superstring in 2T-physics," Phys. Rev. **D70** (2004) 104022, [arXiv:hep-th/0407239].

[8] I. Bars, "Twistors and 2T-physics," AIP Conf. Proc. **767** (2005) 3 , [arXiv:hep-th/0502065].

[9] F. Cachazo, P. Svrcek and E. Witten, " MHV vertices and tree amplitudes in gauge theory", JHEP **0409** (2004) 006 [arXiv:hep-th/0403047]; " Twistor space structure of one-loop amplitudes in gauge theory", JHEP **0410** (2004) 074 [arXiv:hep-th/0406177]; "Gauge theory amplitudes in twistor space and holomorphic anomaly", JHEP **0410** (2004) 077 [arXiv:hep-th/0409245].

[10] For a review of Super Yang-Mills computations and a complete set of references see: F.Cachazo and P.Svrcek, "Lectures on twistor strings and perturbative Yang-Mills theory," PoS **RTN2005** (2005) 004, [arXiv:hep-th/0504194].

[11] I. Bars, " 2T physics formulation of superconformal dynamics relating to twistors and supertwistors," Phys. Lett. B **483**, 248 (2000) [arXiv:hep-th/0004090].

[12] I. Bars and M. Picon, in preparation.

[13] Z. Perj´es, Rep. Math. Phys. **12**, 193 (1977).

[14] L.P. Hughston, "Twistors and Particles", Lecture Notes in Physics **97**, Springer Verlag, Berlin (1979).

[15] P.A. Tod, Rep. Math. Phys. **11**, 339 (1977).

[16] S. Fedoruk and V.G. Zima, "Bitwistor formulation of massive spinning particle," arXiv:hep-th/0308154.

[17] J.A.de Azcarraga, A.Frydryszak, J.Lukierski and C.Miquel-Espanya, "Massive relativistic particle model with spin from free two-twistor dynamics and its quantization," arXiv:hep-th/0510161.

[18] M. Cederwall, "AdS twistors for higher spin theory," AIP Conf. Proc. 767 (2005) 96 , [arXiv:hep-th/0412222].

[19] I. Bars, B. Orcal, and M. Picon, in preparation.

[20] I. Bars, C. Deliduman and O. Andreev, " Gauged Duality, Conformal Symmetry and Spacetime with Two Times" , Phys. Rev. **D58** (1998) 066004 [arXiv:hep-th/9803188]. For reviews of subsequent work see: I. Bars, " Two-Time Physics" , in the Proc. of the 22nd Intl. Colloq. on Group Theoretical Methods in Physics, Eds. S. Corney at. al., World Scientific 1999, [arXiv:hep-th/9809034]; " Survey of two-time physics," Class. Quant. Grav. **18**, 3113 (2001) [arXiv:hep-th/0008164]; " 2T-physics 2001," AIP Conf. Proc. **589** (2001), pp.18-30; AIP Conf. Proc. **607** (2001), pp.17-29 [arXiv:hep-th/0106021].

[21] I. Bars, "Conformal symmetry and duality between free particle, H-atom and harmonic oscillator", Phys. Rev. **D58** (1998) 066006 [arXiv:hep-th/9804028]; "Hidden Symmetries, $\mathrm{AdS}_d \times \mathrm{S}^n$, and the lifting of one-time physics to two-time physics", Phys. Rev. **D59** (1999) 045019 [arXiv:hep-th/9810025].

[22] I. Bars, "Two time physics with gravitational and gauge field backgrounds", Phys. Rev. **D62**, 085015 (2000) [arXiv:hep-th/0002140]; I. Bars and C. Deliduman, " High spin gauge fields and two time physics", Phys. Rev. **D64**, 045004 (2001) [arXiv:hep-th/0103042].

[23] I. Bars, " Hidden 12-dimensional structures in AdS_5 x S^5 and M^4 x R^6 supergravities," Phys. Rev. D **66**, 105024 (2002) [arXiv:hep-th/0208012]. " A mysterious zero in $\mathrm{AdS}_(5)$ x S^5 supergravity," Phys. Rev. D **66**, 105023 (2002) [arXiv:hep-th/0205194].

[24] I. Bars and C. Deliduman, "Gauge symmetry in phase space with spin: a basis for conformal symmetry and duality among many interactions", Phys. Rev. **D58** (1998)106004, [arXiv:hep-th/9806085]. For additional quantum properties of the spinning theory in covariant quantization, including interactions, see the first reference in [25].

[25] I. Bars, " Two-time physics in Field Theory" , Phys. Rev. D **62**, 046007 (2000), [arXiv:hep-th/0003100]; "U*(1,1) noncommutative gauge theory as the foundation of 2T-physics in field theory," Phys. Rev. D **64**, 126001 (2001) [arXiv:hep-th/0106013]; I. Bars and S. J. Rey, " Noncommutative Sp(2,R) gauge

theories: A field theory approach to two-time physics," Phys. Rev. D **64**, 046005 (2001) [arXiv:hep-th/0104135].

[26] M.A. Vasiliev, " Higher spin superalgebras in any dimension and their representations" , arXiv:hep-th/0404124.

[27] I. Bars and B. Orcal, in preparation.